Discrete Mathematics 309 (2009) 2020–2041

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Enumerating S_n by associated transpositions and linear extensions of finite posets

Alessandro Conflitti

Fakultät für Mathematik, Universität Wien, Nordbergstraße 15, A-1090 Wien, Austria

ARTICLE INFO

Article history:
Received 17 January 2005
Received in revised form 30 November 2007
Accepted 4 April 2008
Available online 22 May 2008

Keywords:
Generalized Eulerian polynomials
Statistics over the symmetric group
Partial linear extensions of posets

ABSTRACT

We define a family of statistics over the symmetric group S_n indexed by subsets of the transpositions, and we study the corresponding generating functions. We show that they have many interesting combinatorial properties. In particular we prove that any poset of size n corresponds to a subset of transpositions of S_n, and that the generating function of the corresponding statistic includes partial linear extensions of such a poset. We prove equidistribution results, and we explicitly compute the associated generating functions for several classes of subsets.

1. Overview

In this paper we study some combinatorial and enumerative problems on S_n, the symmetric group of n elements.

Given S_n, one of the most studied objects is its *Eulerian polynomial* $A_n(X) = \sum_{k=1}^{n} A(n,k) X^k$, whose coefficients, called *Eulerian numbers*, count the number of $\sigma \in S_n$ such that σ has $k-1$ descents (see below for definitions). These objects have been widely studied, see e.g. [16,18,21,24,32–34,36,38,45,50,52,53,55] and the references therein, as well as [6,7,15,27–31,40,43,44] for similar combinatorial results. For any $n \geq 1$, the Eulerian polynomial $A_n(X)$ is closely related (in a way which will be explained below) to a particular generating set of S_n, as a group, which is a very special subset of the set of transpositions T_n.

In this paper, we generalize the concept of Eulerian polynomial by considering any subset $\mathcal{T}$ of T_n, and a corresponding polynomial $\Phi_{\mathcal{T}}^{(n)}(X)$ which is an analogue of $A_n(X)$.

Given $n \geq 2$ and $\mathcal{T} \subset T_n$, it is possible to associate a partial order $\preceq_{\mathcal{T}}$ (i.e. indexed by the subset $\mathcal{T}$) on the set $[n]$, so that the corresponding polynomial $\Phi_{\mathcal{T}}^{(n)}(X)$ enumerates partial linear extensions of the poset $([n], \preceq_{\mathcal{T}})$. Moreover any partial order relation $\preceq$ on $[n]$ can be obtained as $\preceq_{\mathcal{T}}$ for some $\mathcal{T} \subset T_n$.

We show that for any $\mathcal{T} \subset T_n$, $\Phi_{\mathcal{T}}^{(n)}(X)$ is symmetric and it has no internal zeros. We define the mirrored set $\mathcal{T}_M$, the order $\operatorname{ord}(\mathcal{T})$ and the normal form $\mathcal{T}_N$, of $\mathcal{T}$ and we prove that knowledge of $\Phi_{\mathcal{T}_N}^{(\operatorname{ord}(\mathcal{T}))}(X)$ or $\Phi_{\mathcal{T}_M}^{(n)}(X)$ gives knowledge of $\Phi_{\mathcal{T}}^{(n)}(X)$.

Moreover, we study equidistribution problems, namely we investigate conditions for different subsets $\mathcal{T}_1 \neq \mathcal{T}_2$ to have the same associated polynomial $\Phi_{\mathcal{T}_1}^{(n)}(X) = \Phi_{\mathcal{T}_2}^{(n)}(X)$. Finally we define some interesting classes of subsets $\mathcal{T} \subset T_n$ for which we can explicitly compute the polynomial $\Phi_{\mathcal{T}}^{(n)}(X)$, and we study some questions about unimodality and log-concavity of $\Phi_{\mathcal{T}}^{(n)}(X)$.

E-mail addresses: alessandro.conflitti@univie.ac.at, conflitt@mat.uc.pt.

0012-365X/$ – see front matter
doi:10.1016/j.disc.2008.04.014

2. Notations and preliminaries

In this section we collect some definitions, notations and results that will be used in the following.

For $x \in \mathbb{R}$ we let $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$ and $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$; for $n \in \mathbb{N}$ we let $[n] = \{t \in \mathbb{N} : 1 \leq t \leq n\} = \{1, \ldots, n\}$, and $[0] = \emptyset$, and we let $[\pm n] = \{t \in \mathbb{N} \setminus \{0\} : |t| \leq n\} = \{\pm 1, \ldots, \pm n\}$. Given $n, m \in \mathbb{N} \setminus \{0\}$, $n \leq m$, we let $[n, m] = [m] \setminus [n-1]$. We let S_n be the set of all bijections $\sigma : [n] \longrightarrow [n]$. If $\sigma \in S_n$ then we write $\sigma = [a_1, \ldots, a_n]$ to mean that $\sigma(j) = a_j$ for $j \in [n]$. Sometimes we also write σ in *disjoint cycle form* and we usually omit writing the 1-cycles of σ. Given $\sigma, \tau \in S_n$ we let $\sigma\tau = \sigma \circ \tau$ (composition of functions) so that, for example $(1, 2)(2, 3) = (1, 2, 3)$. The cardinality of a set $\mathcal{X}$ will be denoted by $\#\mathcal{X}$. For two sets $\mathcal{X}, \mathcal{Y}$ we denote with $\mathcal{X} \uplus \mathcal{Y}$ the *disjoint union* of $\mathcal{X}$ and $\mathcal{Y}$, and with $\mathcal{X} \setminus \mathcal{Y} = \{x : x \in \mathcal{X},\ x \notin \mathcal{Y}\}$ the difference set. If $A \subset \mathbb{R}$ we write $A = \{a_1, \ldots, a_r\}_<$ to mean that $A = \{a_1, \ldots, a_r\}$ and $a_1 < \cdots < a_r$.

A sequence $(a_0, \ldots, a_n)$ of real numbers is *unimodal* if there exists an index $0 \leq k \leq n$ such that $a_j \leq a_{j+1}$ for $j = 0, \ldots, k-1$ and $a_j \geq a_{j+1}$ for $j = k, \ldots, n-1$. It is said to be *log-concave* if $a_j^2 > a_{j-1}a_{j+1}$ for $j \in [n-1]$. It is *symmetric* if $a_j = a_{n-j}$ for $j = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$. It has no *internal zeros* if there do not exist three indices $0 \leq i < j < k \leq n$ such that $a_i a_k \neq 0$ and $a_j = 0$.

It is easy to verify that a strictly positive log-concave sequence is unimodal, though this is not guaranteed for a generic log-concave sequence, (take e.g. $(1, 1, 0, 0, 1, 1)$), and that if the sequence a_j is strictly positive the condition of log-concavity is equivalent to the seemingly stronger condition

$$a_{k-j}a_{k+j} \geq a_{k-j'}a_{k+j'}$$

for all $0 \leq j \leq j' \leq \min\{k, n-k\}$.

A polynomial $\sum_{j=0}^{n} a_j X^j$ is *unimodal* (respectively, *log-concave*, *symmetric*, with no *internal zeros*) if the sequence $(a_0, \ldots, a_n)$ has the corresponding property.

We refer to [48,19] (and the references therein) for comprehensive surveys about unimodal and log-concave sequences.

It is well known that if $\sum_{j=0}^{n} a_j X^j$ is a polynomial with non-negative coefficients and with only real zeros, then the sequence $(a_0, \ldots, a_n)$ is unimodal, log-concave and with no internal zeros. If $f(X)$ is a symmetric polynomial then there is a unique $n \in \mathbb{N}$ such that $X^n f\left(\frac{1}{X}\right) = f(X)$. If $f(X), g(X)$ are two log-concave (respectively symmetric and unimodal) polynomials with non-negative coefficients then the polynomial $f(X) \cdot g(X)$ is also a log-concave (respectively symmetric and unimodal) polynomial with non-negative coefficients, although this is not guaranteed for non-symmetric, unimodal polynomials, see [41].

We follow [5,37,49] for combinatorics and poset notations and terminology, and [12,17,22,26,39,42,51] for comprehensive references about Coxeter groups. In particular, for any $\sigma = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ we say that $j \in [n-1]$ is a *descent* of σ if $a_j > a_{j+1}$. We say that a pair $(i, j) \in [n] \times [n]$ is an *inversion* of σ if $i < j$ and $a_i > a_j$. We denote by $\mathrm{des}(\sigma)$ (respectively, $\mathrm{inv}(\sigma)$) the number of descents (respectively, inversions) of σ.

For any $n \geq 2$ we denote

$$\mathcal{E}_n = \{(j, j+1) : j \in [n-1]\}$$

the *classical generating set* of S_n, and

$$T_n = \{(i, j) : 1 \leq i < j \leq n\}$$

the *reflection set* of S_n.

It is well known that

$$D(\sigma) = \{\tau \in \mathcal{E}_n : \mathrm{inv}(\sigma\tau) < \mathrm{inv}(\sigma)\}$$

is the *descent set* of $\sigma \in S_n$, and

$$I(\sigma) = \{\tau \in T_n : \mathrm{inv}(\sigma\tau) < \mathrm{inv}(\sigma)\}$$

is the *inversion set* of $\sigma \in S_n$, thus $\mathrm{des}(\sigma) = \#D(\sigma)$ and $\mathrm{inv}(\sigma) = \#I(\sigma)$.

For any $\mathcal{T} \subset T_n$ we define

$$I_{\mathcal{T}}(\sigma) = \{\tau \in \mathcal{T} : \mathrm{inv}(\sigma\tau) < \mathrm{inv}(\sigma)\} = \mathcal{T} \bigcap I(\sigma)$$

and

$$l_{\mathcal{T}}(\sigma) = \#I_{\mathcal{T}}(\sigma)$$

for any $\sigma \in S_n$,

$$\mathcal{C}_{\mathcal{T}}^{(n)}(j) = \{\sigma \in S_n : l_{\mathcal{T}}(\sigma) = j\},$$

and

$$\Phi_{\mathcal{T}}^{(n)}(X) = \sum_{\sigma \in S_n} X^{l_{\mathcal{T}}(\sigma)} = \sum_{j=0}^{\#\mathcal{T}} \mathcal{F}_{\mathcal{T}}^{(n)}(j) X^j \in \mathbb{N}[X],$$

where $\mathcal{F}_{\mathcal{T}}^{(n)}(j) = \#\mathcal{C}_{\mathcal{T}}^{(n)}(j)$.

If $\mathcal{T} = T_n$ then $\Phi_{T_n}^{(n)}(X)$ is the well-known *MacMahon (Poincaré) polynomial*:

Proposition 2.1. *For any $n \geq 2$,*

$$\sum_{\sigma \in S_n} X^{\mathrm{inv}(\sigma)} = \prod_{k=1}^{n-1} \left(\sum_{j=0}^{k} X^j \right)$$

holds.

For a proof see Corollary 1.3.10 of [49] or Corollary 4.3 of this paper.

We refer to [30,31,35,43,44] for combinatorial results about $\Phi_{T_n}^{(n)}(X)$.

Definition 2.2. For any $n \geq 2$ we define

$$\digamma_n(X) = \Phi_{\mathcal{E}_n}^{(n)}(X).$$

We have that

$$A_n(X) = X \cdot \digamma_n(X) = \sum_{\sigma \in S_n} X^{1+\mathrm{des}(\sigma)}$$

is the nth *Eulerian polynomial*, which has been widely studied, see e.g. [16,18,21,24,32–34,36,38,45,50,52,53,55] and the references therein, as well as [6,7,15,27–29,40] for similar combinatorial results.

In particular, the following result holds:

Theorem 2.3. *For any $n \geq 2$, the nth Eulerian polynomial $A_n(X)$ is unimodal and log-concave.*

See [16,34,50] for combinatorial proofs.

We always assume that S_n is partially ordered by (strong) *Bruhat order*. We recall (see e.g. [12, Chap. 2] or [42, Section 5.9]) that this means that if $u, v \in S_n$, $u < v$, if and only if there exist $r \in \mathbb{N}$ and $t_1, \ldots, t_r \in T_n$ such that

$$v = ut_1 \cdots t_r,$$
$$\mathrm{inv}(ut_1 \cdots t_{j+1}) > \mathrm{inv}(ut_1 \cdots t_j) \quad \text{for any } j \in [r-1].$$

It is easy to see that $[n, \ldots, 1]$ is the maximum element in the poset S_n with the (strong) Bruhat order.

3. General results

In this section we prove some general results about $\Phi_{\mathcal{T}}^{(n)}(X)$ which hold for any $\mathcal{T} \subset T_n$.

Proposition 3.1. *For any $n \geq 2$ and any $\mathcal{T} \subset T_n$*

$$\mathcal{F}_{\mathcal{T}}^{(n)}(j) > 0$$

for all $j = 0, \ldots, \#\mathcal{T}$.

Proof. We prove the statement by induction on n.

It is quite easy to see that if $n \leq 4$ the desired result holds; suppose that the statement is true for n.

Let $\mathcal{T} \subset T_{n+1}$; we set $t = \#\mathcal{T}$,

$$\mathcal{H}_1 = \{(a, b) \in \mathcal{T} : b \in [n]\},$$

$\mathcal{H}_2 = \mathcal{T} \setminus \mathcal{H}_1$, and $\#\mathcal{H}_2 = k$.

If $k \leq j \leq t$ recalling that $\mathcal{H}_1 \subset T_n \subset T_{n+1}$ by the inductive hypothesis there exists $\tau \in S_n$ such that $\tau \in \mathcal{C}_{\mathcal{H}_1}^{(n)}(j-k)$; then the permutation

$$\sigma(\nu) = \begin{cases} \tau(\nu) + 1 & \text{if } \nu \in [n] \\ 1 & \text{if } \nu = n+1 \end{cases}$$

belongs to $\mathcal{C}_{\mathcal{T}}^{(n+1)}(j)$, so the desired result follows.

If $k > 0$ then we write

$$\mathcal{H}_2 = \{(a_j, n+1) : j \in [k]\}$$

with

$$1 \leq a_1 < \cdots < a_k \leq n.$$

Obviously $[1, \dots, n+1] \in \mathcal{C}_{\mathcal{T}}^{(n+1)}(0)$ and for any $1 \le j \le k$ we let $\sigma \in S_{n+1}$ be

$$\sigma(\nu) = \begin{cases} \nu & \text{if } \nu \in [a_{k-j+1} - 1] \\ \nu + 1 & \text{if } \nu \in [a_{k-j+1}, n] \\ a_{k-j+1} & \text{if } \nu = n+1 \end{cases}$$

then $\sigma \in \mathcal{C}_{\mathcal{T}}^{(n+1)}(j)$, so the desired result follows for any j. □

Proposition 3.2. *For any $n \ge 2$ and any $\mathcal{T} \subset T_n$,*

$$\Phi_{\mathcal{T}}^{(n)}(X) = X^{\#\mathcal{T}}\, \Phi_{\mathcal{T}}^{(n)}\left(\frac{1}{X}\right)$$

holds.

Proof. Let $f : S_n \longrightarrow S_n$ be defined by

$$f([\sigma(1), \dots, \sigma(n)]) = [n+1-\sigma(1), \dots, n+1-\sigma(n)].$$

Obviously f is an involution, and $I(f(\sigma)) = T_n \setminus I(\sigma)$, hence $I_{\mathcal{T}}(f(\sigma)) = \mathcal{T} \setminus I_{\mathcal{T}}(\sigma)$ for any $\mathcal{T} \subset T_n$. □

If $\mathcal{B} \subset T_n$ is a base, viz. a minimal generating set, of S_n, then obviously $\#\mathcal{B} = n-1$ and we can strengthen Proposition 3.1 giving a non-trivial lower bound for any coefficient of $\Phi_{\mathcal{B}}^{(n)}(X)$.

We need the following definition.

Definition 3.3. Let $n \ge 2$ and $\mathcal{H} \subset T_n$: we define the *transitive closure* $\overline{\mathcal{H}}$ of $\mathcal{H}$ as the minimal $\overline{\mathcal{H}} \subset T_n$ such that

- $\mathcal{H} \subset \overline{\mathcal{H}}$,
- if $(a, b) \in \overline{\mathcal{H}}$ and $(b, c) \in \overline{\mathcal{H}}$ then $(a, c) \in \overline{\mathcal{H}}$, for any $1 \le a < b < c \le n$.

For example, $\overline{\mathcal{E}_n} = T_n$.

The following result is well known.

Proposition 3.4. *Let $n \ge 2$ and $\mathcal{H} \subset T_n$: there exists $\sigma \in S_n$ such that $I(\sigma) = \mathcal{H}$ if and only if $\mathcal{H}$ is transitively closed, i.e. $\mathcal{H} = \overline{\mathcal{H}}$.*

Instead of taking the whole inversion set of S_n, consider a generic base $\mathcal{B}$: then we prove that the analogue of Proposition 3.4 for the inversions with respect to $\mathcal{B}$ is always true for any $\mathcal{H} \subset \mathcal{B}$.

Theorem 3.5. *Let $n \ge 2$ and $\mathcal{B} \subset T_n$ be a base of S_n; for any $\mathcal{H} \subset \mathcal{B}$ there exists $\sigma \in S_n$ such that*

$$I_{\mathcal{B}}(\sigma) = \mathcal{H}.$$

Proof. Consider $\overline{\mathcal{H}}$, the transitive closure of $\mathcal{H}$: then from Proposition 3.4 there exists $\sigma \in S_n$ such that $I(\sigma) = \overline{\mathcal{H}}$. We claim that

$$\overline{\mathcal{H}} \bigcap \mathcal{B} = \mathcal{H},$$

because obviously $\mathcal{H} \subset \overline{\mathcal{H}} \bigcap \mathcal{B}$ and there is nothing else since $\mathcal{B}$ is a base, thus if there exist $1 \le a < b < c \le n$ such that $(a, b) \in \mathcal{B}$ and $(b, c) \in \mathcal{B}$ then $(a, c) \notin \mathcal{B}$.

Therefore $I_{\mathcal{B}}(\sigma) = \mathcal{H}$. □

The following result follows from Theorem 3.5.

Corollary 3.6. *Let $n \ge 2$ and $\mathcal{B} \subset T_n$ be a base of S_n; then for any $t = 0, \dots, n-1$*

$$\mathcal{F}_{\mathcal{B}}^{(n)}(t) \ge \binom{n-1}{t}$$

holds. □

We recall the following well-known definition, see e.g. [49].

Definition 3.7. Let $(P, \preceq)$ be a finite poset, with $n = \#P$. Then a *linear extension* of P is an order-preserving bijection $f : P \xrightarrow{\sim} [n]$, that is $x \preceq y$ implies $f(x) \le f(y)$ for any $x, y \in P$.

We now show that the constant term of $\Phi_{\mathcal{T}}^{(n)}(X)$ counts the number of linear extensions of a certain poset of size n, associated to $\mathcal{T}$.

Definition 3.8. Let $n \ge 2$ and $\mathcal{T} \subset T_n$; we equip the set $[n]$ with an order relation $\preceq_{\mathcal{T}}$ defined in the following way: for any $1 \le a \le b \le n$, $a \preceq_{\mathcal{T}} b$ if and only if either $a = b$ or $a < b$ and $(a, b) \in \overline{\mathcal{T}}$, the transitive closure of $\mathcal{T}$.

It is clear that for any $n \geq 2$, any poset of size n is isomorphic to $([n], \preceq_{\mathcal{T}})$ for some $\mathcal{T} \subset T_n$.
The following result is immediate with these definitions.

Proposition 3.9. *For any $n \geq 2$ and any $\mathcal{T} \subset T_n$, $\Phi_{\mathcal{T}}^{(n)}(0) = \mathcal{F}_{\mathcal{T}}^{(n)}(0)$ equals the number of linear extensions of the poset $([n], \preceq_{\mathcal{T}})$.* $\square$

More generally, if a permutation $\sigma \in \mathcal{C}_{\mathcal{T}}^{(n)}(j)$ then j is a measure of "how far" σ is from being a linear extension of $([n], \preceq_{\mathcal{T}})$.
Now we define the position set, the order and the normal form of a subset $\mathcal{T} \subset T_n$ and we prove a couple of results about $\Phi_{\mathcal{T}}^{(n)}(X)$ which are helpful for the practical computation of these polynomials, see Section 5.

Definition 3.10. Let $n \geq 2$ and $\mathcal{T} = \{(a_1, b_1), \ldots, (a_k, b_k)\} \subset T_n$.
We define the *position set* of $\mathcal{T}$ as

$$\mathrm{Pos}(\mathcal{T}) = \{a_1, \ldots, a_k, b_1, \ldots, b_k\} \subset [n],$$

viz. the subset of $[n]$ which is involved in $\mathcal{T}$.
We define the *order* of $\mathcal{T}$ by

$$\mathrm{ord}(\mathcal{T}) = \#\,\mathrm{Pos}(\mathcal{T}).$$

Let $\mathrm{ord}(\mathcal{T}) = t$ and $\mathrm{Pos}(\mathcal{T}) = \{\omega_1, \ldots, \omega_t\}_<$; we define the *normal form* $\mathcal{T}_N \subset T_t$ of $\mathcal{T}$ in the following way: for any $1 \leq i < j \leq t$ $(i, j) \in \mathcal{T}_N$ if and only if $(\omega_i, \omega_j) \in \mathcal{T}$.

Essentially, the normal form of $\mathcal{T} \subset T_n$ squeezes $\mathcal{T}$ as much as possible while holding its structure.
For example, if

$$\mathcal{T} = \{(3, 7), (4, 9), (5, 7), (11, 15), (20, 21), (20, 25), (22, 24)\} \subset T_{31},$$

then

$$\mathcal{T}_N = \{(1, 4), (2, 5), (3, 4), (6, 7), (8, 9), (8, 12), (10, 11)\} \subset T_{12}.$$

The following result shows that it is enough to compute $\Phi_{\mathcal{T}}^{(n)}(X)$ for subsets $\mathcal{T}$ which are in normal form.

Proposition 3.11. *Let $n \geq 2$ and $\mathcal{T} \subset T_n$. Then*

$$\Phi_{\mathcal{T}}^{(n)}(X) = \frac{n!}{t!} \Phi_{\mathcal{T}_N}^{(t)}(X),$$

where $t = \mathrm{ord}(\mathcal{T})$ and $\mathcal{T}_N$ is the normal form of $\mathcal{T}$.

Proof. Let $\mathcal{T} = \{(a_1, b_1), \ldots, (a_k, b_k)\} \subset T_n$ and

$$\mathrm{Pos}(\mathcal{T}) = \{\omega_1, \ldots, \omega_t\}_<.$$

We consider $X = \{x_1, \ldots, x_t\} \subset [n]$ and

$$A_X = \{\sigma \in S_n : \{\sigma(\omega_1), \ldots, \sigma(\omega_t)\} = X\}.$$

Then for any $0 \leq j \leq k = \#\mathcal{T}$ we have

$$\#\{\sigma \in A_X : l_{\mathcal{T}}(\sigma) = j\} = (n-t)!\mathcal{F}_{\mathcal{T}_N}^{(t)}(j).$$

Therefore $\#\{\sigma \in A_X : l_{\mathcal{T}}(\sigma) = j\}$ does not depend on the choice of X, and we have

$$\mathcal{F}_{\mathcal{T}}^{(n)}(j) = \sum_{\substack{X \subset [n] \\ \#X = t}} \#\{\sigma \in A_X : l_{\mathcal{T}}(\sigma) = j\}$$
$$= \binom{n}{t} \#\{\sigma \in A_X : l_{\mathcal{T}}(\sigma) = j\} = \binom{n}{t}(n-t)!\mathcal{F}_{\mathcal{T}_N}^{(t)}(j)$$

as desired. $\square$

This result shows that for any $n \geq 3$ there are $\mathcal{T}_1, \mathcal{T}_2 \subset T_n$ such that $\mathcal{T}_1 \neq \mathcal{T}_2$ and $\Phi_{\mathcal{T}_1}^{(n)}(X) = \Phi_{\mathcal{T}_2}^{(n)}(X)$. In fact, it is enough that $(\mathcal{T}_1)_N = (\mathcal{T}_2)_N$.
In the following we examine closely this topic, see Proposition 3.15, Section 4, and Proposition 5.1.
For any $n \geq 2$ and any $\mathcal{T} \subset T_n$, we have that $\mathrm{ord}(\mathcal{T}) \leq n$, and $\mathrm{ord}(\mathcal{T}) = n$ if and only if $\mathcal{T}$ is in normal form. Now we count the number of $\mathcal{T} \subset T_n$ which have maximum order, i.e. such that $\mathcal{T} = \mathcal{T}_N$. This is the same as the number of graphs on $[n]$ with no isolated vertices.

Proposition 3.12. *For any $n \geq 2$,*

$$\#\{\mathcal{T} \subset T_n : \mathrm{ord}\,(\mathcal{T}) = n\} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} 2^{\binom{n-j}{2}}.$$

Proof. Obviously, for any $2 \leq k \leq n$ we have that $\# T_k = \binom{k}{2}$, so $\#\{\mathcal{T} \subset T_k\} = 2^{\binom{k}{2}}$. For any $h \in [n]$, let

$$A_h = \{\mathcal{T} \subset T_n : h \notin \mathrm{Pos}\,(\mathcal{T})\};$$

then by inclusion–exclusion we have

$$\#\{\mathcal{T} \subset T_n : \mathrm{ord}\,(\mathcal{T}) = n\} = 2^{\binom{n}{2}} - \sum_{j=1}^{n} (-1)^{j-1} \sum_{\substack{\mathcal{I} \subset [n] \\ \#\mathcal{I} = j}} \# \left(\bigcap_{h \in \mathcal{I}} A_h \right).$$

Since

$$\# \left(\bigcap_{h \in \mathcal{I}} A_h \right) = \left\{ \mathcal{T} \subset T_n : \mathcal{I} \bigcap \mathrm{Pos}\,(\mathcal{T}) = \emptyset \right\} = 2^{\binom{n-\#\mathcal{I}}{2}},$$

the desired result follows. □

Now we consider $\mathcal{T} \subset T_n$ such that $\mathcal{T} = \mathcal{T}_1 \biguplus \mathcal{T}_2$. At first it would be tempting to suppose that it is possible to compute $\Phi^{(n)}_{\mathcal{T}}(X)$ from $\Phi^{(n)}_{\mathcal{T}_1}(X)$ and $\Phi^{(n)}_{\mathcal{T}_2}(X)$, but unfortunately the hypothesis $\mathcal{T}_1 \bigcap \mathcal{T}_2 = \emptyset$ is too weak. For example,

$$\Phi^{(4)}_{\{(1,2)\}}(X) = \Phi^{(4)}_{\{(1,3)\}}(X) = \Phi^{(4)}_{\{(2,3)\}}(X) = \Phi^{(4)}_{\{(3,4)\}}(X) = 12 + 12X,$$

but

$$\Phi^{(4)}_{\{(1,2),(1,3)\}}(X) = 8 + 8X + 8X^2,$$
$$\Phi^{(4)}_{\{(1,2),(2,3)\}}(X) = 4 + 16X + 4X^2,$$
$$\Phi^{(4)}_{\{(1,2),(3,4)\}}(X) = 6 + 12X + 6X^2.$$

Nevertheless, this is possible, by replacing the hypothesis $\mathcal{T}_1$ and $\mathcal{T}_2$ disjoint with the stronger condition $\mathrm{Pos}\,(\mathcal{T}_1)$ and $\mathrm{Pos}\,(\mathcal{T}_2)$ disjoint.

Proposition 3.13. *Let $n \geq 4$ and let $\mathcal{T}_1, \ldots, \mathcal{T}_r \subset T_n$ be such that*

$$\mathrm{Pos}\,(\mathcal{T}_i) \bigcap \mathrm{Pos}\,(\mathcal{T}_j) = \emptyset$$

for any $i \neq j$. Then

$$\Phi^{(n)}_{\biguplus_{k=1}^{r} \mathcal{T}_k}(X) = n! \prod_{k=1}^{r} \left(\frac{1}{t_k!} \Phi^{(t_k)}_{(\mathcal{T}_k)_N}(X) \right),$$

where $t_k = \mathrm{ord}\,(\mathcal{T}_k)$ for any $k \in [r]$.

Proof. We prove the statement by induction on r.

If $r = 2$ by Proposition 3.11 we can assume without loss of generality that $\mathcal{T} = \mathcal{T}_1 \biguplus \mathcal{T}_2$ is in normal form, i.e. $\mathrm{Pos}\,(\mathcal{T}_1) \biguplus \mathrm{Pos}\,(\mathcal{T}_2) = [n]$, so $t_1 + t_2 = n$.

For any $0 \leq z \leq \#\mathcal{T} = \#\mathcal{T}_1 + \#\mathcal{T}_2$, we have that

$$\mathcal{F}^{(n)}_{\mathcal{T}}(z) = \sum_{j=0}^{z} \# \left(\mathcal{C}^{(n)}_{\mathcal{T}_1}(j) \bigcap \mathcal{C}^{(n)}_{\mathcal{T}_2}(z-j) \right). \tag{1}$$

As in the proof of Proposition 3.11, let

$$\mathrm{Pos}\,(\mathcal{T}_1) = \{\omega_1, \ldots, \omega_{t_1}\}_<$$

and consider $X = \{x_1, \ldots, x_{t_1}\} \subset [n]$, and

$$A_X = \{\sigma \in S_n : \{\sigma(\omega_1), \ldots, \sigma(\omega_{t_1})\} = X\}.$$

Then for any $0 \leq j \leq z \leq \#\mathcal{T}$ we have

$$\begin{aligned} \#\{\sigma \in A_X : l_{\mathcal{T}_1}(\sigma) = j, l_{\mathcal{T}_2}(\sigma) = z - j\} &= \# \left(A_X \bigcap \mathcal{C}^{(n)}_{\mathcal{T}_1}(j) \bigcap \mathcal{C}^{(n)}_{\mathcal{T}_2}(z-j) \right) \\ &= \mathcal{F}^{(t_1)}_{\mathcal{T}_1}(j) \cdot \mathcal{F}^{(t_2)}_{\mathcal{T}_2}(z-j), \end{aligned}$$

because in order to have $l_{\mathcal{T}_1}(\sigma) = j$ there are $\mathcal{F}_{\mathcal{T}_1}^{(t_1)}(j)$ ways to arrange $x_1, \ldots, x_{t_1}$ in positions $\mathrm{Pos}(\mathcal{T}_1) = \{\omega_1, \ldots, \omega_{t_1}\}$ and in order to have $l_{\mathcal{T}_2}(\sigma) = z-j$ there are $\mathcal{F}_{\mathcal{T}_2}^{(t_2)}(z-j)$ ways to arrange $[n] \setminus \{x_1, \ldots, x_{t_1}\}$ in positions $\mathrm{Pos}(\mathcal{T}_2) = [n] \setminus \{\omega_1, \ldots, \omega_{t_1}\}$.

Therefore $\#\left(A_X \bigcap \mathcal{C}_{\mathcal{T}_1}^{(n)}(j) \bigcap \mathcal{C}_{\mathcal{T}_2}^{(n)}(z-j)\right)$ does not depend on the choice of X, and we have

$$\begin{aligned}\#\left(\mathcal{C}_{\mathcal{T}_1}^{(n)}(j) \bigcap \mathcal{C}_{\mathcal{T}_2}^{(n)}(z-j)\right) &= \sum_{\substack{X \subset [n] \\ \#X = t_1}} \#\left(A_X \bigcap \mathcal{C}_{\mathcal{T}_1}^{(n)}(j) \bigcap \mathcal{C}_{\mathcal{T}_2}^{(n)}(z-j)\right) \\ &= \frac{n!}{t_1! t_2!} \mathcal{F}_{\mathcal{T}_1}^{(t_1)}(j)\, \mathcal{F}_{\mathcal{T}_2}^{(t_2)}(z-j)\end{aligned}$$

because $t_1 + t_2 = n$, and from (1) the desired result follows.

If $r > 2$ let $\mathcal{Z}_1 = \biguplus_{k=1}^{r-1} \mathcal{T}_k$; then we have $\mathrm{Pos}(\mathcal{Z}_1) \bigcap \mathrm{Pos}(\mathcal{T}_r) = \emptyset$ and from the case $r = 2$ we get

$$\Phi_{\biguplus_{k=1}^{r} \mathcal{T}_k}^{(n)}(X) = \frac{n!}{z! t_r!} \Phi_{(\mathcal{Z}_1)_N}^{(z)}(X)\, \Phi_{(\mathcal{T}_r)_N}^{(t_r)}(X),$$

where $z = \mathrm{ord}(\mathcal{Z}_1)$. By induction on r the desired result follows. □

Definition 3.14. Let $n \geq 2$ and $\mathcal{T} \subset T_n$. We define the *mirrored* $\mathcal{T}$, $\mathcal{T}_M$, in the following way:

$$(i, j) \in \mathcal{T}_M \quad \text{if and only if } (n + 1 - j, n + 1 - i) \in \mathcal{T}.$$

We define $\mathcal{T} \subset T_n$ to be *self-mirrored* if $\mathcal{T} = \mathcal{T}_M$.

For example, if $n = 4$ then any $\mathcal{T} \subset T_4$ such that $\#\mathcal{T} = 2$ and $\mathrm{ord}(\mathcal{T}) = 4$ is self-mirrored.

We remark that for any $n \geq 2$ and any $\mathcal{T} \subset T_n$,

$$(\mathcal{T}_M)_M = \mathcal{T}$$

and

$$(\mathcal{T}_N)_M = (\mathcal{T}_M)_N.$$

Proposition 3.15. *For any $n \geq 2$ and $\mathcal{T} \subset T_n$,*

$$\Phi_{\mathcal{T}}^{(n)}(X) = \Phi_{\mathcal{T}_M}^{(n)}(X).$$

Proof. Consider the involution $f : S_n \longrightarrow S_n$ defined by

$$f([\sigma(1), \ldots, \sigma(n)]) = [\sigma(n), \ldots, \sigma(1)].$$

It is clear that for any $j = 0, \ldots, t = \#\mathcal{T} = \#\mathcal{T}_M$ $\sigma \in \mathcal{C}_{\mathcal{T}}^{(n)}(j)$ if and only if $f(\sigma) \in \mathcal{C}_{\mathcal{T}_M}^{(n)}(t-j)$. But by Proposition 3.2, $\mathcal{F}_{\mathcal{T}_M}^{(n)}(t-j) = \mathcal{F}_{\mathcal{T}_M}^{(n)}(j)$ and the desired result follows. □

Obviously for any $n \geq 2$ and $\mathcal{T} \subset T_n$ we have

$$(T_n \setminus \mathcal{T})_M = T_n \setminus \mathcal{T}_M,$$

therefore:

Corollary 3.16. *For any $n \geq 2$ and $\mathcal{T} \subset T_n$,*

$$\Phi_{T_n \setminus \mathcal{T}}^{(n)}(X) = \Phi_{T_n \setminus \mathcal{T}_M}^{(n)}(X). \quad \square$$

4. Equidistribution results

In this section we investigate some conditions for different subsets $\mathcal{T}_1 \neq \mathcal{T}_2$ with $(\mathcal{T}_1)_N \neq (\mathcal{T}_2)_N \neq ((\mathcal{T}_1)_N)_M$ to have the same associated polynomial, i.e. $\Phi_{\mathcal{T}_1}^{(n)}(X) = \Phi_{\mathcal{T}_2}^{(n)}(X)$.

We remark that equidistribution results and questions are widely studied in combinatorics, see e.g. [1,3,4,10,30,31,43,44] and the references therein.

Definition 4.1. Let $n \geq 2$ and $\mathcal{T} \subset T_n$; we define

$$m(\mathcal{T}) = \min(\mathrm{Pos}(\mathcal{T})) \quad \text{and} \quad M(\mathcal{T}) = \max(\mathrm{Pos}(\mathcal{T})).$$

Let $k \in \mathbb{Z}$ be such that $-m(\mathcal{T}) < k \leq n - M(\mathcal{T})$; we define the *k-shifted* $\mathcal{T}$, $\mathcal{T}_k^+$, in the following way:

$$(i, j) \in \mathcal{T} \quad \text{if and only if } (i + k, j + k) \in \mathcal{T}_k^+.$$

Lemma 4.2. *Let $n \geq 2$ and $\mathcal{T} \subset T_n$: we let*

$$\mathcal{T}_+ = \mathcal{T} \biguplus \{(j, n+1) : j \in [n]\} \subset T_{n+1}$$

and

$$\mathcal{T}_- = \mathcal{T}_1^+ \biguplus \{(1, j) : j \in [2, n+1]\} \subset T_{n+1}.$$

Then

$$\Phi^{(n+1)}_{\mathcal{T}_+}(X) = \Phi^{(n+1)}_{\mathcal{T}_-}(X) = \Phi^{(n)}_{\mathcal{T}}(X) \cdot \sum_{j=0}^{n} X^j.$$

Proof. For any $k \geq 2$ if $\mathcal{H}_1, \mathcal{H}_2 \subset T_k$ are such that $\mathcal{H}_1 \bigcap \mathcal{H}_2 = \emptyset$ then it is clear that

$$I_{\mathcal{H}_1 \uplus \mathcal{H}_2}(\sigma) = I_{\mathcal{H}_1}(\sigma) \biguplus I_{\mathcal{H}_2}(\sigma)$$

for any $\sigma \in S_k$, hence $l_{\mathcal{H}_1 \uplus \mathcal{H}_2}(\sigma) = l_{\mathcal{H}_1}(\sigma) + l_{\mathcal{H}_2}(\sigma)$.

Therefore

$$\Phi^{(n+1)}_{\mathcal{T}_+}(X) = \sum_{\sigma \in S_{n+1}} X^{l_{\mathcal{T}_+}(\sigma)} = \sum_{j=1}^{n+1} \sum_{\substack{\sigma \in S_{n+1} \\ \sigma(n+1)=j}} X^{l_{\mathcal{T}}(\sigma)} \cdot X^{n+1-j} = \Phi^{(n)}_{\mathcal{T}}(X) \cdot \sum_{j=0}^{n} X^j$$

and similarly for $\Phi^{(n+1)}_{\mathcal{T}_-}(X)$. $\square$

Note that Lemma 4.2 implies Proposition 2.1: in fact

Corollary 4.3. *For any $n \geq 2$,*

$$\sum_{\sigma \in S_n} X^{\mathrm{inv}(\sigma)} = \prod_{k=1}^{n-1} \left(\sum_{j=0}^{k} X^j \right).$$

Proof. Obviously if $n = 2$ the result holds, and for any $n \geq 3$ we have that $T_{n+1} = (T_n)_+$, and we can apply Lemma 4.2. $\square$

Theorem 4.4. *Let $n \geq 3$, $\mathcal{H} \subset T_n$, and $-m(\mathcal{H}) < k \leq n - M(\mathcal{H})$. Then*

$$\Phi^{(n)}_{T_n \setminus \mathcal{H}}(X) = \Phi^{(n)}_{T_n \setminus \mathcal{H}_k^+}(X).$$

Proof. Clearly it is enough to prove the statement for $k = 1$, so assume $M(\mathcal{H}) < n$. Then

$$\left((T_n \setminus \mathcal{H}_1^+) \setminus \{(1, j) : j \in [2, n]\} \right) = \left((T_n \setminus \mathcal{H}) \setminus \{(j, n) : j \in [n-1]\} \right)_1^+$$

and the result follows from Lemma 4.2 (applied to $\mathcal{T} = (T_n \setminus \mathcal{H}) \setminus \{(j, n) : j \in [n-1]\}$). $\square$

The following Corollary is immediate.

Corollary 4.5. *Let $n \geq 3$ and*

$$\mathcal{T}_j = T_n \setminus \{(a_j, b_j)\},$$

$j \in \{1, 2\}$, be such that $b_1 - a_1 = b_2 - a_2 \in [n-1]$. Then

$$\Phi^{(n)}_{\mathcal{T}_1}(X) = \Phi^{(n)}_{\mathcal{T}_2}(X). \quad \square$$

Note that $\mathcal{T} = T_n \setminus \{(a, b)\}$ if and only if $\#\mathcal{T} = (\#T_n) - 1 = \frac{n(n-1)}{2} - 1$. Therefore there are $\binom{\#T_n}{\#\mathcal{T}} = \#T_n = \frac{n(n-1)}{2}$ different such subsets $\mathcal{T}$ but by Corollary 4.5 there are only at most $n-1$ distinct polynomials $\Phi^{(n)}_{\mathcal{T}}(X)$. Note also that Theorem 4.4 and Corollary 4.5 are the best possible: in fact if $n = 4$ and $\mathcal{T}_j = T_4 \setminus \{(a_j, b_j)\}$ for $j \in [2]$ and $b_1 - a_1 \neq b_2 - a_2$ then $\Phi^{(4)}_{\mathcal{T}_1}(X) \neq \Phi^{(4)}_{\mathcal{T}_2}(X)$.

Corollary 4.6. *For any $n \geq 3$, any $r \in [n-1]$, and any $a \in [n-r]$*

$$\Phi^{(n)}_{T_n \setminus \{(a, a+r)\}}(X) = \Phi^{(r+1)}_{T_{r+1} \setminus \{(1, r+1)\}}(X) \cdot \prod_{k=r+1}^{n-1} \left(\sum_{j=0}^{k} X^j \right).$$

In particular, for any $a \in [n-1]$

$$\Phi^{(n)}_{T_n \setminus \{(a, a+1)\}}(X) = 2 \prod_{k=2}^{n-1} \left(\sum_{j=0}^{k} X^j \right).$$

Proof. From Corollary 4.5 we have $\Phi^{(n)}_{T_n\setminus\{(a,a+r)\}}(X) = \Phi^{(n)}_{T_n\setminus\{(1,r+1)\}}(X)$ for any $a \in [n-r]$. Obviously $T_n \setminus \{(1, r+1)\} = (T_{n-1} \setminus \{(1, r+1)\})_+$ for any $n \geq r+2$, and the desired result follows from Lemma 4.2. $\square$

Proposition 4.7. *Let $n \geq 2$ and $\mathcal{T} \subset T_n$ be such that there exist $1 \leq a < b \leq n$ such that*

$$\left(\{(a,j) : j \in [b] \setminus [a]\} \biguplus \{(k,b) : k \in [b-1] \setminus [a]\}\right) \bigcap \mathcal{T} = \emptyset.$$

We define

$$\mathcal{T}' = \{(a,b)\,(i,j)\,(a,b) : (i,j) \in \mathcal{T}\}.$$

Then

$$\Phi^{(n)}_{\mathcal{T}}(X) = \Phi^{(n)}_{\mathcal{T}'}(X)\,.$$

Proof. Consider the involution $\varphi : S_n \xrightarrow{\sim} S_n$ defined by $\varphi(\sigma) = \sigma(a,b) = (\sigma(a), \sigma(b))\,\sigma$.

It is easy to see that $l_{\mathcal{T}}(\sigma) = l_{\mathcal{T}'}(\varphi(\sigma))$ for any $\sigma \in S_n$, therefore

$$\Phi^{(n)}_{\mathcal{T}}(X) = \sum_{\sigma\in S_n} X^{l_{\mathcal{T}}(\sigma)} = \sum_{\sigma\in S_n} X^{l_{\mathcal{T}'}(\sigma)} = \Phi^{(n)}_{\mathcal{T}'}(X)\,. \quad \square$$

Corollary 4.8. *Let $n \geq 2$, $\mathcal{T} \subset T_n$ and $a \in [n-1]$ be such that $(a, a+1) \notin \mathcal{T}$. Then*

$$\Phi^{(n)}_{\mathcal{T}}(X) = \Phi^{(n)}_{\mathcal{T}'}(X)\,,$$

where $\mathcal{T}' = \{(a,a+1)\,(i,j)\,(a,a+1) : (i,j) \in \mathcal{T}\}$.

Corollary 4.9. *Let $n \geq 3$ and $\mathcal{T} = T_n \setminus \{(\alpha,\alpha+1), (\alpha,\beta)\}$ for some $1 \leq \alpha < \alpha+1 < \beta \leq n$. Then*

$$\Phi^{(n)}_{\mathcal{T}}(X) = \Phi^{(n)}_{\mathcal{T}'}(X)\,,$$

where $\mathcal{T}' = T_n \setminus \{(\alpha,\alpha+1), (\alpha+1,\beta)\}$.

Proof. Apply Proposition 4.7 with $a = \alpha$ and $b = \alpha + 1$. $\square$

In some sense it is possible to generalize Theorem 4.4.

Definition 4.10. For any $n \geq 2$, let $1 \leq \xi_1 < \xi_2 \leq n$. We define $\mathcal{T} \subset T_n$ to be a (ξ_1, ξ_2)-*screen* if the following conditions are satisfied:

(1) $(i,j) \in \mathcal{T}$ for any $\xi_1 \leq i < j \leq \xi_2$,
(2) if there exists $1 \leq i_0 < \xi_1$ such that $(i_0, r) \in \mathcal{T}$ for some $\xi_1 \leq r \leq \xi_2$, then $(i_0, r) \in \mathcal{T}$ for all $\xi_1 \leq r \leq \xi_2$,
(3) if there exists $\xi_2 < j_0 \leq n$ such that $(r, j_0) \in \mathcal{T}$ for some $\xi_1 \leq r \leq \xi_2$, then $(r, j_0) \in \mathcal{T}$ for all $\xi_1 \leq r \leq \xi_2$.

For example, T_n is the only $(1, n)$-screen.

Note that if $\mathcal{T} \subset T_n$ is a (ξ_1, ξ_2)-screen, then it is also a (ϖ_1, ϖ_2)-screen for any $\xi_1 \leq \varpi_1 < \varpi_2 \leq \xi_2$.

Furthermore, if n is sufficiently large, there exists $\mathcal{T} \subset T_n$ which is simultaneously the (ξ_1, ξ_2)-screen and (ξ_3, ξ_4)-screen for some $1 \leq \xi_1 < \xi_2 < \xi_3 < \xi_4 \leq n$, and such that $\mathcal{T}$ is not a (ξ_1, ξ_4)-screen.

For example, consider an increasing sequence $1 = c_1 < \cdots < c_{2r} = n$ of $2r$ elements such that $c_{j+1} - c_j \geq 3$ for any $j \in [2r-1]$, and let

$$\mathcal{T} = T_n \setminus \biguplus_{k=1}^{r} \{(c_{2k-1}, c_{2k})\}.$$

Then $\mathcal{T}$ is a $(c_j + 1, c_{j+1} - 1)$-screen for any $j \in [2r-1]$, and there are no wider screens inside $\mathcal{T}$.

Proposition 4.11. *For any $n \geq 2$ and any $1 \leq \xi_1 < \xi_2 \leq n$,*

$$\#\{\mathcal{T} \subset T_n : \mathcal{T} \text{ is a } (\xi_1, \xi_2)\text{-screen}\} = 2^{\binom{n-\xi_2+\xi_1}{2}}.$$

Proof. Note first that for $\mathcal{T}$ to be a (ξ_1, ξ_2)-screen, no constraints are required for (i,j) such that $1 \leq i < j < \xi_1$, $1 \leq i < \xi_1 < \xi_2 < j \leq n$, and $\xi_2 < i < j \leq n$. Furthermore, for any $i \in [\xi_1 - 1]$ (respectively $j \in [n] \setminus [\xi_2]$) then either $\biguplus_{k=\xi_1}^{\xi_2}\{(i,k)\} \subset \mathcal{T}$ or $\biguplus_{k=\xi_1}^{\xi_2}\{(i,k)\} \bigcap \mathcal{T} = \emptyset$, (respectively either $\biguplus_{k=\xi_1}^{\xi_2}\{(k,j)\} \subset \mathcal{T}$ or $\biguplus_{k=\xi_1}^{\xi_2}\{(k,j)\} \bigcap \mathcal{T} = \emptyset$).

Therefore identifying $[\xi_1, \xi_2]$ as a single element, we get a bijection

$$\{\mathcal{T} \subset T_n : \mathcal{T} \text{ is a } (\xi_1, \xi_2)\text{-screen}\} \xrightarrow{\sim} \{\mathcal{T} \subset T_{n-\xi_2+\xi_1}\},$$

and the result follows. $\square$

Definition 4.12. For any $n \geq 2$ and $A \subset [n]$ we define

$$\mathcal{G}(A) = \{(i, j) \in A \times A : i < j\}.$$

For example, $\mathcal{G}([n]) = T_n$, and

$$\mathcal{G}(\{1, 3, 5, 7\}) = \{(1, 3), (1, 5), (1, 7), (3, 5), (3, 7), (5, 7)\}$$

for any $n \geq 7$.

Lemma 4.13. *Let $n \geq 2$, and let $\mathcal{T} \subset T_n$ be a (ξ_1, ξ_2)-screen for some $1 \leq \xi_1 < \xi_2 \leq n$. For any $\sigma \in S_n$ let $\xi_1 \leq a_\sigma < b_\sigma \leq \xi_2$, $1 \leq u_\sigma < v_\sigma \leq n$. We define*

$$\Omega_\sigma = \{a_\sigma \leq j \leq b_\sigma : u_\sigma \leq \sigma(j) \leq v_\sigma\},$$

and let $f_\sigma \in S_n$ be any permutation such that $f_\sigma(j) = \sigma(j)$ if $j \notin \Omega_\sigma$. Then $l_{\mathcal{T}}(\sigma) = l_{\mathcal{T}}(f_\sigma)$ if and only if $l_{\mathcal{G}(\Omega_\sigma)}(\sigma) = l_{\mathcal{G}(\Omega_\sigma)}(f_\sigma)$.

Proof. We have that $\mathcal{G}(\Omega_\sigma) \subset \mathcal{T}$, then $I_{\mathcal{T}}(\sigma) = I_{\mathcal{T} \setminus \mathcal{G}(\Omega_\sigma)}(\sigma) \biguplus I_{\mathcal{G}(\Omega_\sigma)}(\sigma)$ for any $\sigma \in S_n$, therefore the statement of the Lemma is equivalent to the existence of a bijection $I_{\mathcal{T} \setminus \mathcal{G}(\Omega_\sigma)}(\sigma) \stackrel{\sim}{\longleftrightarrow} I_{\mathcal{T} \setminus \mathcal{G}(\Omega_\sigma)}(f_\sigma)$.

By hypothesis $f_\sigma(j) = \sigma(j)$ for any $j \notin \Omega_\sigma$, hence for any $\gamma \in [a_\sigma, b_\sigma] \setminus \Omega_\sigma$ there are only two possibilities:

$$\sigma(\gamma) > v_\sigma \geq \max_{j \in \Omega_\sigma} \sigma(j) = \max_{j \in \Omega_\sigma} f_\sigma(j),$$

or

$$\sigma(\gamma) < u_\sigma \leq \min_{j \in \Omega_\sigma} \sigma(j) = \min_{j \in \Omega_\sigma} f_\sigma(j).$$

Therefore for any $r_1, r_2 \in \Omega_\sigma$ such that $r_1 < \gamma < r_2$, we have $(r_1, \gamma) \notin I_{\mathcal{T} \setminus \mathcal{G}(\Omega_\sigma)}(\sigma) \bigcup I_{\mathcal{T} \setminus \mathcal{G}(\Omega_\sigma)}(f_\sigma)$, and $(\gamma, r_2) \in I_{\mathcal{T} \setminus \mathcal{G}(\Omega_\sigma)}(\sigma) \bigcap I_{\mathcal{T} \setminus \mathcal{G}(\Omega_\sigma)}(f_\sigma)$ for the case $\gamma > v_\sigma$, whereas $(r_1, \gamma) \in I_{\mathcal{T} \setminus \mathcal{G}(\Omega_\sigma)}(\sigma) \bigcap I_{\mathcal{T} \setminus \mathcal{G}(\Omega_\sigma)}(f_\sigma)$, and $(\gamma, r_2) \notin I_{\mathcal{T} \setminus \mathcal{G}(\Omega_\sigma)}(\sigma) \bigcup I_{\mathcal{T} \setminus \mathcal{G}(\Omega_\sigma)}(f_\sigma)$ for the case $\gamma < u_\sigma$.

The remaining pairs $(h, k) \in \mathcal{T} \setminus \mathcal{G}(\Omega_\sigma)$ to check are:

- $1 \leq h < k \leq n$ with $h, k \notin \Omega_\sigma$,
- $h \in [a_\sigma - 1]$ and $k \in \Omega_\sigma$,
- $h \in \Omega_\sigma$ and $k \in [b_\sigma + 1, n]$.

In all these cases it is easy to see that $(h, k) \in I_{\mathcal{T} \setminus \mathcal{G}(\Omega_\sigma)}(\sigma)$ if and only if $\left(f_\sigma^{-1}(\sigma(h)), f_\sigma^{-1}(\sigma(k))\right) \in I_{\mathcal{T} \setminus \mathcal{G}(\Omega_\sigma)}(f_\sigma)$, because $\mathcal{T}$ is a (ξ_1, ξ_2)-screen, thus a (a_σ, b_σ)-screen, and if $f_\sigma(x) \neq \sigma(x)$ then $x \in \Omega_\sigma \subset [a_\sigma, b_\sigma]$. Therefore the desired result follows. $\square$

Theorem 4.14. *Let $n \geq 2$ and $\mathcal{T} \subset T_n$ be a (ξ_1, ξ_2)-screen for some $1 \leq \xi_1 < \xi_2 \leq n$. Let $\xi_1 \leq a_1 < b_1 \leq \xi_2$ and $\xi_1 \leq a_2 < b_2 \leq \xi_2$ be such that $b_1 - a_1 = b_2 - a_2$, and $\mathcal{T}_1 = \mathcal{T} \setminus \{(a_1, b_1)\}$, $\mathcal{T}_2 = \mathcal{T} \setminus \{(a_2, b_2)\}$.*

Then

$$\Phi_{\mathcal{T}_1}^{(n)}(X) = \Phi_{\mathcal{T}_2}^{(n)}(X).$$

Proof. Obviously if $k = b_1 - a_1 = b_2 - a_2 = \xi_2 - \xi_1$ then $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T} \setminus \{(\xi_1, \xi_2)\}$ and the result is trivial, so we can assume $k \in [\xi_2 - \xi_1 - 1]$, and clearly it is enough to prove the statement for $\mathcal{T}_1 = \mathcal{T} \setminus \{(a, a + k)\}$ and $\mathcal{T}_2 = \mathcal{T} \setminus \{(a + 1, a + k + 1)\}$, for any $\xi_1 \leq a \leq \xi_2 - (k + 1)$.

If $\mathcal{H}_1, \mathcal{H}_2 \subset T_n$ are such that $\mathcal{H}_1 \bigcap \mathcal{H}_2 = \emptyset$ then it is clear that

$$I_{\mathcal{H}_1 \bigcup \mathcal{H}_2}(\sigma) = I_{\mathcal{H}_1}(\sigma) \biguplus I_{\mathcal{H}_2}(\sigma)$$

for any $\sigma \in S_n$, therefore we can write $S_n = B_{00} \biguplus B_{10} \biguplus B_{01} \biguplus B_{11}$, where

$$\begin{aligned}
B_{00} &= \{\sigma \in S_n : (a, a + k) \notin I_{\mathcal{T}}(\sigma), (a + 1, a + k + 1) \notin I_{\mathcal{T}}(\sigma)\} \\
&= \{\sigma \in S_n : l_{\mathcal{T}}(\sigma) = l_{\mathcal{T}_1}(\sigma) = l_{\mathcal{T}_2}(\sigma)\}, \\
B_{10} &= \{\sigma \in S_n : (a, a + k) \in I_{\mathcal{T}}(\sigma), (a + 1, a + k + 1) \notin I_{\mathcal{T}}(\sigma)\} \\
&= \{\sigma \in S_n : l_{\mathcal{T}}(\sigma) = l_{\mathcal{T}_1}(\sigma) + 1 = l_{\mathcal{T}_2}(\sigma)\}, \\
B_{01} &= \{\sigma \in S_n : (a, a + k) \notin I_{\mathcal{T}}(\sigma), (a + 1, a + k + 1) \in I_{\mathcal{T}}(\sigma)\} \\
&= \{\sigma \in S_n : l_{\mathcal{T}}(\sigma) = l_{\mathcal{T}_1}(\sigma) = l_{\mathcal{T}_2}(\sigma) + 1\}, \\
B_{11} &= \{\sigma \in S_n : (a, a + k) \in I_{\mathcal{T}}(\sigma), (a + 1, a + k + 1) \in I_{\mathcal{T}}(\sigma)\} \\
&= \{\sigma \in S_n : l_{\mathcal{T}}(\sigma) = l_{\mathcal{T}_1}(\sigma) + 1 = l_{\mathcal{T}_2}(\sigma) + 1\}.
\end{aligned}$$

Thus we have

$$\Phi_{\mathcal{T}_1}^{(n)}(X) = \sum_{\sigma\in B_{00}} X^{l_{\mathcal{T}}(\sigma)} + \frac{1}{X}\sum_{\sigma\in B_{11}} X^{l_{\mathcal{T}}(\sigma)} + \frac{1}{X}\sum_{\sigma\in B_{10}} X^{l_{\mathcal{T}}(\sigma)} + \sum_{\sigma\in B_{01}} X^{l_{\mathcal{T}}(\sigma)},$$
$$\Phi_{\mathcal{T}_2}^{(n)}(X) = \sum_{\sigma\in B_{00}} X^{l_{\mathcal{T}}(\sigma)} + \frac{1}{X}\sum_{\sigma\in B_{11}} X^{l_{\mathcal{T}}(\sigma)} + \sum_{\sigma\in B_{10}} X^{l_{\mathcal{T}}(\sigma)} + \frac{1}{X}\sum_{\sigma\in B_{01}} X^{l_{\mathcal{T}}(\sigma)}.$$

Hence $\Phi_{\mathcal{T}_1}^{(n)}(X) = \Phi_{\mathcal{T}_2}^{(n)}(X)$ if and only if

$$\sum_{\sigma\in B_{10}} X^{l_{\mathcal{T}}(\sigma)} = \sum_{\sigma\in B_{01}} X^{l_{\mathcal{T}}(\sigma)},$$

which is equivalent to the existence of a bijection $f : B_{10} \xrightarrow{\sim} B_{01}$ such that $l_{\mathcal{T}}(\sigma) = l_{\mathcal{T}}(f(\sigma))$ for any $\sigma \in B_{10}$.

It is easy to see that if $k = 1$, i.e. $\mathcal{T}_1 = \mathcal{T} \setminus \{(a, a+1)\}$ and $\mathcal{T}_2 = \mathcal{T} \setminus \{(a+1, a+2)\}$, then the map

$$f(\sigma) = \begin{cases} \sigma(a, a+2, a+1) & \text{if } \sigma(a) < \sigma(a+2) \\ \sigma(a, a+1, a+2) & \text{if } \sigma(a) > \sigma(a+2) \end{cases}$$

is the desired bijection, so the result follows.

If $2 \le k \le \xi_2 - \xi_1 - 1$ define $\mathcal{U}_k = \{(a, a+1), (a+k, a+k+1)\} \subset T_n$; then

$$S_n = \mathcal{C}_{\mathcal{U}_k}^{(n)}(0) \biguplus \mathcal{C}_{\mathcal{U}_k}^{(n)}(1) \biguplus \mathcal{C}_{\mathcal{U}_k}^{(n)}(2),$$

where

$$\mathcal{C}_{\mathcal{U}_k}^{(n)}(0) = \{\tau \in S_n : \tau(a) < \tau(a+1),\ \tau(a+k) < \tau(a+k+1)\}$$
$$\begin{aligned}\mathcal{C}_{\mathcal{U}_k}^{(n)}(1) \ &= \ \{\tau \in S_n : \tau(a) > \tau(a+1),\ \tau(a+k) < \tau(a+k+1)\} \\ &\biguplus \ \{\tau \in S_n : \tau(a) < \tau(a+1),\ \tau(a+k) > \tau(a+k+1)\}\end{aligned}$$
$$\mathcal{C}_{\mathcal{U}_k}^{(n)}(2) = \{\tau \in S_n : \tau(a) > \tau(a+1),\ \tau(a+k) > \tau(a+k+1)\}.$$

Consider the involution

$$g : B_{10} \bigcap \mathcal{C}_{\mathcal{U}_k}^{(n)}(1) \xrightarrow{\sim} B_{01} \bigcap \mathcal{C}_{\mathcal{U}_k}^{(n)}(1)$$

defined by $\sigma \longmapsto \sigma(a+k, a+k+1)(a, a+1)$.

Then $l_{\mathcal{T}}(\sigma) = l_{\mathcal{T}}(g(\sigma))$ for any $\sigma \in B_{10} \bigcap \mathcal{C}_{\mathcal{U}_k}^{(n)}(1)$. So in order to get $\Phi_{\mathcal{T}_1}^{(n)}(X) = \Phi_{\mathcal{T}_2}^{(n)}(X)$ it is enough to exhibit two bijections

$$\begin{aligned}&\Phi : B_{10} \bigcap \mathcal{C}_{\mathcal{U}_k}^{(n)}(0) \xrightarrow{\sim} B_{01} \bigcap \mathcal{C}_{\mathcal{U}_k}^{(n)}(0) \\ &\sigma \longmapsto \Phi_\sigma\end{aligned}$$

and

$$\begin{aligned}&\Psi : B_{10} \bigcap \mathcal{C}_{\mathcal{U}_k}^{(n)}(2) \xrightarrow{\sim} B_{01} \bigcap \mathcal{C}_{\mathcal{U}_k}^{(n)}(2) \\ &\sigma \longmapsto \Psi_\sigma\end{aligned}$$

such that $l_{\mathcal{T}}(\sigma) = l_{\mathcal{T}}(\Phi_\sigma)$ for any $\sigma \in B_{10} \bigcap \mathcal{C}_{\mathcal{U}_k}^{(n)}(0)$ and $l_{\mathcal{T}}(\sigma) = l_{\mathcal{T}}(\Psi_\sigma)$ for any $\sigma \in B_{10} \bigcap \mathcal{C}_{\mathcal{U}_k}^{(n)}(2)$.

Note that if $\sigma \in B_{10} \bigcap \mathcal{C}_{\mathcal{U}_k}^{(n)}(0)$ then

$$\sigma(a+k) < \sigma(a) < \sigma(a+1) < \sigma(a+k+1) \tag{2}$$

and if $\sigma \in B_{10} \bigcap \mathcal{C}_{\mathcal{U}_k}^{(n)}(2)$ then

$$\sigma(a+1) < \sigma(a+k+1) < \sigma(a+k) < \sigma(a). \tag{3}$$

Now we define Φ.

For any $\sigma \in B_{10} \bigcap \mathcal{C}_{\mathcal{U}_k}^{(n)}(0)$ we define the following sets:

$$\Omega_\sigma = \{a \le j \le a+k+1 : \sigma(a+k) \le \sigma(j) \le \sigma(a+k+1)\}$$

and

$$\mathcal{R}_\sigma = \Omega_\sigma \setminus \{a, a+1, a+k, a+k+1\}.$$

Let $r = \#\mathcal{R}_\sigma$; we have to distinguish three different cases.

If $r = 0$ then we define $\Phi_\sigma = \sigma(a+1, a+k+1)(a, a+k)$.

If $r = 1$ let $\mathcal{R}_\sigma = \{\alpha\}$: then we define

$$\Phi_\sigma = \begin{cases} \sigma\,(a, a+k+1, a+1, \alpha, a+k) & \text{if } \sigma\,(a+k) < \sigma\,(\alpha) < \sigma\,(a) \\ \sigma\,(a+1, a+k+1)\,(a, a+k) & \text{if } \sigma\,(a) < \sigma\,(\alpha) < \sigma\,(a+1) \\ \sigma\,(a, \alpha, a+k+1, a+1, a+k) & \text{if } \sigma\,(a+1) < \sigma\,(\alpha) < \sigma\,(a+k+1)\,. \end{cases}$$

If $r \geq 2$ let $\mathcal{R}_\sigma = \{\alpha_1, \ldots, \alpha_r\}_<$.
Then we define

$$\Phi_\sigma = \begin{cases} \sigma\tau^2 & \text{if } \sigma\,(\alpha_{r-1}) < \sigma\,(\alpha_r) \\ \sigma\,(a, a+1)\,(\alpha_{r-1}, \alpha_r)\,\tau^2 & \text{if } \sigma\,(\alpha_{r-1}) > \sigma\,(\alpha_r)\,, \end{cases}$$

where $\tau = (a, a+1, \alpha_1, \ldots, \alpha_r, a+k, a+k+1)$.

Note that if $s = \#\{j \in [n] : \Phi_\sigma\,(j) \neq \sigma\,(j)\}$ then τ^2 is the product of two $\frac{s}{2}$ cycles if $s \equiv 0 \pmod 2$, whereas τ^2 is an s cycle if $s \equiv 1 \pmod 2$.

It is not hard to check that, in all cases, $\Phi_\sigma \in B_{01} \bigcap \mathcal{C}^{(n)}_{\mathcal{U}_k}(0)$.

In order to check that $l_{\mathcal{T}}\,(\sigma) = l_{\mathcal{T}}\,(\Phi_\sigma)$ for any $\sigma \in B_{10} \bigcap \mathcal{C}^{(n)}_{\mathcal{U}_k}(0)$, we let

$$Z_1 = \mathcal{R}_\sigma \biguplus \{a, a+1\}$$
$$Z_2 = \mathcal{R}_\sigma \biguplus \{a+k, a+k+1\},$$

and

$$\Delta_1 = \mathcal{G}\,(Z_1)\,,$$
$$\Delta_2 = \mathcal{G}\,(\Omega_\sigma) \setminus \mathcal{G}\,(Z_1)\,,$$
$$\Gamma_1 = \mathcal{G}\,(Z_2)\,,$$
$$\Gamma_2 = \mathcal{G}\,(\Omega_\sigma) \setminus \mathcal{G}\,(Z_2)\,,$$

so

$$\mathcal{G}\,(\Omega_\sigma) = \Delta_1 \biguplus \Delta_2 = \Gamma_1 \biguplus \Gamma_2.$$

For any $\sigma \in B_{10} \bigcap \mathcal{C}^{(n)}_{\mathcal{U}_k}(0)$, from (2) and the definition of Φ, it is easy to see that $l_{\Delta_1}\,(\sigma) = l_{\Gamma_1}\,(\Phi_\sigma)$ and by direct counting that $l_{\Delta_2}\,(\sigma) = l_{\Gamma_2}\,(\Phi_\sigma) = r+2$, therefore $l_{\mathcal{G}(\Omega_\sigma)}\,(\sigma) = l_{\mathcal{G}(\Omega_\sigma)}\,(\Phi_\sigma)$ and from Lemma 4.13 we get $l_{\mathcal{T}}\,(\sigma) = l_{\mathcal{T}}\,(\Phi_\sigma)$.

Analogously, substituting the conditions in (2) with the conditions in (3), we build the map Ψ, which is a 'mirrored' version of the map Φ.

For any $\sigma \in B_{10} \bigcap \mathcal{C}^{(n)}_{\mathcal{U}_k}(2)$ let

$$\Omega^1_\sigma = \{a \leq j \leq a+k+1 : \sigma\,(a+1) \leq \sigma\,(j) \leq \sigma\,(a)\},$$
$$\mathcal{R}^1_\sigma = \Omega^1_\sigma \setminus \{a, a+1, a+k, a+k+1\},$$
$$r_1 = \#\mathcal{R}^1_\sigma.$$

If $r_1 = 0$ then we define $\Psi_\sigma = \sigma\,(a+1, a+k+1)\,(a, a+k)$.

If $r_1 = 1$ let $\mathcal{R}^1_\sigma = \{\beta\}$: then as for α in the case $r = 1$, there are three different subcases:

$$\Psi_\sigma = \begin{cases} \sigma\,(a, a+k, \beta, a+1, a+k+1) & \text{if } \sigma\,(a+1) < \sigma\,(\beta) < \sigma\,(a+k+1) \\ \sigma\,(a+1, a+k+1)\,(a, a+k) & \text{if } \sigma\,(a+k+1) < \sigma\,(\beta) < \sigma\,(a+k) \\ \sigma\,(a, a+k, a+1, a+k+1, \beta) & \text{if } \sigma\,(a+k) < \sigma\,(\beta) < \sigma\,(a)\,. \end{cases}$$

If $r_1 \geq 2$ let $\mathcal{R}^1_\sigma = \{\beta_1, \ldots, \beta_{r_1}\}_<$.
Then we define

$$\Psi_\sigma = \begin{cases} \sigma\rho^2 & \text{if } \sigma\,(\beta_1) > \sigma\,(\beta_2) \\ \sigma\,(\beta_1, \beta_2)\,(a+k, a+k+1)\,\rho^2 & \text{if } \sigma\,(\beta_1) < \sigma\,(\beta_2)\,, \end{cases}$$

where $\rho = (a+k+1, a+k, \beta_{r_1}, \ldots, \beta_1, a+1, a)$.

As for Φ, from Lemma 4.13 and the definition of Ψ, we see that $l_{\mathcal{T}}\,(\sigma) = l_{\mathcal{T}}\,(\Psi_\sigma)$ for any $\sigma \in B_{10} \bigcap \mathcal{C}^{(n)}_{\mathcal{U}_k}(2)$, and the desired result follows. $\square$

Iterating several times Theorem 4.14, we get the following

Theorem 4.15. *For any $n \geq 2$, let $\{\xi_j\}_{j=1}^{2k}$ be a sequence of $2k$ terms for some $k \leq \left\lfloor \frac{n}{2} \right\rfloor$, such that $1 \leq \xi_1 < \cdots < \xi_{2k} \leq n$, and let $\mathcal{T} \subset T_n$ be a (ξ_{2j-1}, ξ_{2j})-screen for any $j \in [k]$. We consider two increasing sequences*

$$1 \leq \alpha_1 < \cdots < \alpha_{2r} \leq n$$
$$1 \leq \beta_1 < \cdots < \beta_{2r} \leq n,$$

both of $2r$ *terms, for some* $k \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor$*, which satisfy the following properties:*

$$\alpha_{2j} - \alpha_{2j-1} = \beta_{2j} - \beta_{2j-1} \quad \textit{for any } j \in [r]\,, \tag{4}$$

and there exist $t_1, \ldots, t_k \in \mathbb{N}$ *such that* $\sum_{i=1}^{k} t_i = r$*, and*

$$\begin{aligned} &1 \leq \xi_1 \leq \alpha_1 < \cdots < \alpha_{2t_1} \leq \xi_2 < \xi_3 \leq \alpha_{1+2t_1} < \cdots < \alpha_{2(t_1+t_2)} \leq \xi_4 \\ &\quad \leq \cdots \leq \xi_{2k-2} < \xi_{2k-1} \leq \alpha_{1+2\sum_{i=1}^{k-1} t_i} < \cdots < \alpha_{2\sum_{i=1}^{k} t_i} \leq \xi_{2k} \leq n, \end{aligned}$$

and analogously for the sequence β*.*

Let

$$\mathcal{T}_1 = \mathcal{T} \setminus \biguplus_{j=1}^{r} \{(\alpha_{2j-1}, \alpha_{2j})\}$$

and

$$\mathcal{T}_2 = \mathcal{T} \setminus \biguplus_{j=1}^{r} \{(\beta_{2j-1}, \beta_{2j})\};$$

then

$$\Phi^{(n)}_{\mathcal{T}_1}(X) = \Phi^{(n)}_{\mathcal{T}_2}(X)\,.$$

Proof. We may clearly assume $t_1 > 0$; and we proceed by induction on r.

Consider

$$\mathcal{A} = \{j \in [t_1 - 1] : \beta_{2j} < \alpha_{2j+1}\}$$

and define

$$j_0 = \begin{cases} \min \mathcal{A} & \text{if } \mathcal{A} \neq \emptyset, \\ t_1 & \text{if } \mathcal{A} = \emptyset. \end{cases}$$

Let

$$\mathcal{T}_{0,1} = \mathcal{T} \setminus \biguplus_{\substack{j=1 \\ j \neq j_0}}^{r} \{(\alpha_{2j-1}, \alpha_{2j})\} = \mathcal{T}_1 \biguplus \{(\alpha_{2j_0-1}, \alpha_{2j_0})\},$$

and we define

$$\mathcal{Z} = \mathcal{T}_{0,1} \setminus \{(\beta_{2j_0-1}, \beta_{2j_0})\}.$$

We claim that $\Phi^{(n)}_{\mathcal{T}_1}(X) = \Phi^{(n)}_{\mathcal{Z}}(X)$.

In fact, let $\varpi_1 = \min\{\alpha_{2j_0-1}, \beta_{2j_0-1}\}$ and $\varpi_2 = \max\{\alpha_{2j_0}, \beta_{2j_0}\}$: then by our construction of j_0 we have that $\mathcal{T}_{0,1}$ is a (ϖ_1, ϖ_2)-screen, because

$$\max\{\xi_1, \alpha_{2j_0-2} + 1\} \leq \varpi_1 < \varpi_2 \leq \min\{\alpha_{2j_0+1} - 1, \xi_2\},$$

where, if $j_0 \in \{1, r\}$, we set $\max\{\xi_1, \alpha_0 + 1\} = \xi_1$ and $\min\{\alpha_{2r+1} - 1, \xi_2\} = \xi_2$.

Taking into account the hypothesis (4), we can apply Theorem 4.14 to get

$$\Phi^{(n)}_{\mathcal{T}_1}(X) = \Phi^{(n)}_{\mathcal{Z}}(X)\,.$$

Hence we have substituted the transposition $(\alpha_{2j_0-1}, \alpha_{2j_0})$ with the corresponding transposition $(\beta_{2j_0-1}, \beta_{2j_0})$.

Now we consider the set

$$\mathcal{W}_0 = \mathcal{T} \setminus \{(\beta_{2j_0-1}, \beta_{2j_0})\}$$

instead of $\mathcal{T}$; we see that it is a (ξ_{2j-1}, ξ_{2j})-screen for any $j \in [k] \setminus \{1\}$, and a $(\xi_1, \beta_{2j_0-1} - 1)$-screen if $\beta_{2j_0-1} \geq \xi_1 + 3$, and a $(\beta_{2j_0} + 1, \xi_2)$-screen if $\beta_{2j_0} \leq \xi_2 - 3$.

Then considering the set $\mathcal{W}_0$ instead of $\mathcal{T}$, the sequence

$$\begin{aligned} &\{\xi_1, \beta_{2j_0-1} - 1 : \text{ if } \beta_{2j_0-1} \geq \xi_1 + 3\} \\ &\biguplus \{\beta_{2j_0} + 1, \xi_2 : \text{ if } \beta_{2j_0} \leq \xi_2 - 3\} \biguplus \{\xi_j\}_{j=3}^{2k} \end{aligned}$$

instead of the sequence $\{\xi_j\}_{j=1}^{2k}$, and the sequences

$$\{\alpha_j : j \in [2r] \setminus \{2j_0 - 1, 2j_0\}\}$$

and

$$\{\beta_j : j \in [2r] \setminus \{2j_0 - 1, 2j_0\}\}$$

instead of the sequences $\{\alpha_j\}_{j=1}^{2r}$ and $\{\beta_j\}_{j=1}^{2r}$ respectively, we have that we can apply the inductive hypothesis. $\square$

We note that in some sense Theorem 4.15 is more general than Theorem 4.4: in fact instead of T_n, i.e. the $(1, n)$-screen, it is allowed to have (ξ_1, ξ_2)-screens, for $1 \le \xi_1 < \xi_2 \le n$, and above all

$$\mathcal{T}_2 = \mathcal{T} \setminus \biguplus_{j=1}^{r} \{(\beta_{2j-1}, \beta_{2j})\}$$

is not in general a k-shifted of

$$\mathcal{T}_1 = \mathcal{T} \setminus \biguplus_{j=1}^{r} \{(\alpha_{2j-1}, \alpha_{2j})\}$$

for some $k \in \mathbb{Z}$.

For example, the following result is immediate from Theorems 4.4 and 4.15.

Corollary 4.16. *Let $n \ge 4$, $1 \le a < b < c < d \le n$, and let*

$$\mathcal{H} = T_n \setminus \{(a, b), (c, d)\} \quad \textit{and} \quad \mathcal{T} = T_n \setminus \{(a, d), (b, c)\}.$$

For any $1 \le a_1 < b_1 < c_1 < d_1 \le n$ such that $b_1 - a_1 = b - a$ and $d_1 - c_1 = d - c$, and any $1 \le a_2 < b_2 < c_2 < d_2 \le n$ such that $d_2 - a_2 = d - a$ and $c_2 - b_2 = c - b$, let

$$\mathcal{H}_1 = T_n \setminus \{(a_1, b_1), (c_1, d_1)\} \quad \textit{and} \quad \mathcal{T}_1 = T_n \setminus \{(a_2, d_2), (b_2, c_2)\}.$$

Then

$$\Phi_{\mathcal{H}}^{(n)}(X) = \Phi_{\mathcal{H}_1}^{(n)}(X)$$

and

$$\Phi_{\mathcal{T}}^{(n)}(X) = \Phi_{\mathcal{T}_1}^{(n)}(X). \quad \square$$

If $n = 7$ and $\mathcal{T}_1, \mathcal{T}_2 \subset T_7$ are such that $\#\mathcal{T}_1 = \#\mathcal{T}_2 = \#T_7 - 2 = 19$ and $\Phi_{\mathcal{T}_1}^{(7)}(X) = \Phi_{\mathcal{T}_2}^{(7)}(X)$ one can check that $\mathcal{T}_1$ and $\mathcal{T}_2$ necessarily satisfy the hypothesis of either Corollary 3.16 or Theorem 4.4 or Corollary 4.9 or Corollary 4.16.

5. Closed formulas for $\Phi_{\mathcal{T}}^{(n)}(X)$

In this section we define some classes of subsets $\mathcal{T} \subset T_n$ and we give closed formulas for their associated polynomials $\Phi_{\mathcal{T}}^{(n)}(X)$. Moreover, we prove some results about $\Phi_{\mathcal{T}}^{(n)}(X)$ when $\#\mathcal{T}$ satisfies certain conditions.

Let $\mathcal{T} \subset T_n$: from the definitions it is immediate that

$$\operatorname{ord}(\mathcal{T}) \le \min\{n, 2\#\mathcal{T}\}.$$

If $\operatorname{ord}(\mathcal{T}) = 2\#\mathcal{T}$, we are able to explicitly compute the polynomial $\Phi_{\mathcal{T}}^{(n)}(X)$.

Proposition 5.1. *Let $n \ge 2$ and let $\mathcal{T} \subset T_n$ be such that* $\operatorname{ord}(\mathcal{T}) = 2\#\mathcal{T}$*. Then*

$$\Phi_{\mathcal{T}}^{(n)}(X) = n! \left(\frac{1+X}{2}\right)^k,$$

where $k = \#\mathcal{T}$.

Proof. $\mathcal{T}$ is such that $\operatorname{ord}(\mathcal{T}) = 2k$, where $k = \#\mathcal{T} \in \left[\left\lfloor \frac{n}{2} \right\rfloor\right]$, so we have

$$\mathcal{T} = \{(a_1, b_1), \ldots, (a_k, b_k)\}$$

and $a_1, \ldots, a_k, b_1, \ldots, b_k$ are all distinct.

If $k = 1$ then $\mathcal{T}_N = \{(1, 2)\}$, $\Phi_{\{(1,2)\}}^{(2)}(X) = 1 + X$, and the desired result follows from Proposition 3.11.

If $k > 1$ the desired result follows from Proposition 3.13. $\square$

For example, let $n = 4$ and $\mathcal{T}_1 = \{(1, 2), (3, 4)\}$, $\mathcal{T}_2 = \{(1, 3), (2, 4)\}$ and $\mathcal{T}_3 = \{(1, 4), (2, 3)\}$. Then $\mathcal{T}_j$, for $j \in [3]$, are in normal form, self-mirrored, and pairwise different, but $\Phi_{\mathcal{T}_1}^{(4)}(X) = \Phi_{\mathcal{T}_2}^{(4)}(X) = \Phi_{\mathcal{T}_3}^{(4)}(X) = 6 + 12X + 6X^2$.

Definition 5.2. Let $n \geq 2$, and $k \in [n-1]$.

We define

$$\mathcal{D}_k = \{(a,b) \in T_n : a \equiv b \, (\mathrm{mod}\, k)\}.$$

We define $\mathcal{T} \subset T_n$ to be a *k-stair* if

$$\mathcal{T} = \{(a,b_1), \ldots, (a,b_k)\}$$

with $1 \leq a < b_1 < \cdots < b_k \leq n$ or

$$\mathcal{T} = \{(a_1,b), \ldots, (a_k,b)\}$$

with $1 \leq a_1 < \cdots < a_k < b \leq n$.

We define $\mathcal{T} \subset T_n$ to be a *k-chain* if

$$\mathcal{T} = \{(a_1,a_2), (a_2,a_3), \ldots, (a_k,a_{k+1})\}$$

with $1 \leq a_1 < \cdots < a_{k+1} \leq n$.

Note that for any $n \geq 2$ there is only one $(n-1)$-chain $\mathcal{T} \subset T_n$, i.e. $\mathcal{T} = \mathcal{E}_n$ the classical generating set of S_n, and the associated polynomial $X \cdot F_n(X)$ is the well-known *Eulerian polynomial*.

Proposition 5.3. *Let $n \geq 2$ and $k \in [n-1]$; then*

$$\Phi^{(n)}_{\mathcal{D}_k}(X) = n! \left[\frac{1}{\left(\left\lfloor \frac{n}{k} \right\rfloor + 1\right)!} \prod_{j=1}^{\left\lfloor \frac{n}{k} \right\rfloor} \left(\sum_{t=0}^{j} X^t \right) \right]^z \left[\frac{1}{\left(\left\lfloor \frac{n}{k} \right\rfloor\right)!} \prod_{j=1}^{\left\lfloor \frac{n}{k} \right\rfloor - 1} \left(\sum_{t=0}^{j} X^t \right) \right]^{k-z},$$

where z is such that $0 \leq z \leq k-1$ and $n \equiv z \, (\mathrm{mod}\, k)$, i.e. $z = n - k\left\lfloor \frac{n}{k} \right\rfloor$.

Proof. Write $\mathcal{D}_k = \biguplus_{j=1}^{k} \mathcal{D}_k(j)$, where

$$\mathcal{D}_k(j) = \{(a,b) \in T_n : a \equiv b \equiv j \, (\mathrm{mod}\, k)\}.$$

It is clear that $\mathrm{Pos}(\mathcal{D}_k(j_1)) \bigcap \mathrm{Pos}(\mathcal{D}_k(j_2)) = \emptyset$ for any $j_1 \neq j_2 \in [k]$, and that for any $j \in [z]$ we have $\mathrm{ord}(\mathcal{D}_k(j)) = \left\lfloor \frac{n}{k} \right\rfloor + 1$ and $(\mathcal{D}_k(j))_N = T_{\left\lfloor \frac{n}{k} \right\rfloor + 1}$, whereas for any $j \in [z+1,k]$ we have $\mathrm{ord}(\mathcal{D}_k(j)) = \left\lfloor \frac{n}{k} \right\rfloor$ and $(\mathcal{D}_k(j))_N = T_{\left\lfloor \frac{n}{k} \right\rfloor}$, therefore from Propositions 3.13 and 2.1 the desired result follows. □

Proposition 5.4. *Let $n \geq 2$, $k \in [n-1]$ and $\mathcal{T} \subset T_n$ be a k-stair. Then*

$$\Phi^{(n)}_{\mathcal{T}}(X) = \frac{n!}{k+1} \sum_{j=0}^{k} X^j.$$

Proof. We consider the case

$$\mathcal{T} = \{(a,b_1), \ldots, (a,b_k)\}$$

with $1 \leq a < b_1 < \cdots < b_k \leq n$; the case $\mathcal{T} = \{(a_1,b), \ldots, (a_k,b)\}$ is completely symmetric.

From Proposition 3.11

$$\Phi^{(n)}_{\mathcal{T}}(X) = \frac{n!}{(k+1)!} \Phi^{(k+1)}_{\mathcal{T}_N}(X), \tag{5}$$

where

$$\mathcal{T}_N = \{(1,\nu) : \nu \in [k+1] \setminus \{1\}\} \subset T_{k+1}.$$

For any $0 \leq j \leq k$, we have that $\sigma \in \mathcal{C}^{(k+1)}_{\mathcal{T}_N}(j)$ if and only if $\sigma(1) = j+1$, therefore $\mathcal{F}^{(k+1)}_{\mathcal{T}_N}(j) = k!$ and from (5) the desired result follows. □

For a k-chain, we have the following result.

Proposition 5.5. *Let $n \geq 2$, $k \in [n-1]$ and $\mathcal{T} \subset T_n$ be a k-chain. Then*

$$\Phi^{(n)}_{\mathcal{T}}(X) = \frac{n!}{(k+1)!} F_{k+1}(X).$$

Proof. We have that $\mathcal{T}_N = \{(1,2), (2,3), \ldots, (k, k+1)\}$, and the desired result follows from Proposition 3.11. $\square$

Note that if $\mathcal{T} \subset T_n$ is such that $\#\mathcal{T} = 2$ then $\mathcal{T}$ is either a 2-stair, or a 2-chain, or $\mathrm{ord}(\mathcal{T}) = 4$.
Now we generalize the concepts of k-chain and k-stair.

Definition 5.6. Let $n \geq 2$, and $1 \leq \nu \leq \mu \leq n$. We define $P_\mu, Q_{\mu,\nu} \subset T_n$ as

$$P_\mu = \{(j,\mu) : j \in [\mu - 1]\} \biguplus \{(\mu, k) : k \in [n] \setminus [\mu]\},$$
$$Q_{\mu,\nu} = \mathcal{E}_\mu \biguplus \{(\nu, k) : k \in [n] \setminus [\mu]\},$$

with the convention $\mathcal{E}_1 = \emptyset$, so $Q_{1,1} = P_1$ and $Q_{n-1,n-1} = Q_{n,j} = \mathcal{E}_n$ for any $j \in [n]$.

Obviously for any $n \geq 2$, $\#P_\mu = \#Q_{\mu,\nu} = n - 1$ for any $\mu \in [n]$ and $\nu \in [\mu]$.
Note that for any $n \geq 2$, $P_1 = Q_{1,1}$ and P_n are the only $n-1$-stairs in T_n, so from Proposition 5.4 we know that

$$\Phi^{(n)}_{P_1}(X) = \Phi^{(n)}_{Q_{1,1}}(X) = \Phi^{(n)}_{P_n}(X) = (n-1)! \sum_{j=0}^{n-1} X^j. \tag{6}$$

From a Coxeter group point of view (see [12,17,22,26,39,42,51]) for any $n \geq 2$, $\mu \in [n]$ and $\nu \in [\mu]$, P_μ and $Q_{\mu,\nu}$ are bases, i.e. minimal generating sets, of S_n exactly as $\mathcal{E}_n$, the classical generating set, see e.g. [47], so in this sense (6), (8) and (13), see below, can be considered as analogues of

$$\Phi^{(n)}_{Q_{n-1,n-1}}(X) = \Phi^{(n)}_{Q_{n,j}}(X) = F_n(X),$$

for any $j \in [n]$.

Proposition 5.7. *For any $n \geq 2$, $\mu \in [n]$ and $\nu \in [\mu]$,*

$$\Phi^{(n)}_{Q_{\mu,\nu}}(0) = \binom{n-\nu}{n-\mu}(n-\mu)! = \prod_{j=0}^{n-\mu-1}(n - \nu - j)$$

holds.

Proof. It is easy to see that $\sigma \in \mathcal{C}^{(n)}_{Q_{\mu,\nu}}(0)$ if and only if $\sigma(j) = j$ for any $j \in [\nu]$,

$$A = \{\sigma(k) : k \in [n] \setminus [\mu]\}$$

is allowed to be any subset of $[n] \setminus [\nu]$ with cardinality $n - \mu$, and

$$\{\sigma(t) : t \in [\mu] \setminus [\nu]\} = [n] \setminus \left([\nu] \biguplus A\right)$$

as ordered sets, i.e. $\sigma(r+1) > \sigma(r)$ for any $r \in [\mu - 1] \setminus [\nu]$.
The desired result follows. $\square$

Corollary 5.8. *For any $n \geq 2$ and any $j \in [n-1]$, there exists $\mathcal{T}_j \subset T_n$ such that*

$$\Phi^{(n)}_{\mathcal{T}_j}(0) = j.$$

Proof. Take $\mathcal{T}_j = Q_{n-1,n-j}$ and the desired result follows from Proposition 5.7. $\square$

Now we investigate the base P_μ explicitly computing $\Phi^{(n)}_{P_\mu}(X)$ for any $\mu \in [n]$.

Theorem 5.9. *For any $n \geq 2$ and $\mu \in [n]$,*

$$\Phi^{(n)}_{P_\mu}(X) = \Phi^{(n)}_{P_{n+1-\mu}}(X), \tag{7}$$

and for any $t = 0, \ldots, n-1$

$$\mathcal{F}^{(n)}_{P_\mu}(t) = (n-\mu)!(\mu-1)! \sum_{j=\max(0,\mu-n+t)}^{\min(t,\mu-1)} \binom{n-\mu+2j-t}{j}\binom{\mu-2j+t-1}{t-j} \tag{8}$$

holds.
Furthermore,

$$\mathcal{F}^{(n)}_{P_\mu}(t) \leq (n-\mu)!(\mu-1)! \sum_{j=0}^{\mu-1} \binom{n-1-j}{\mu-1-j} 2^j = \mathcal{F}^{(n)}_{P_\mu}(t_0) \tag{9}$$

for any $\mu - 1 \leq t_0 \leq n - \mu$.

Proof. Obviously, (7) follows from Proposition 3.15.

Let $t = 0, \ldots, n-1$ and $\sigma \in \mathcal{C}_{P_\mu}^{(n)}(t)$; we set

$$\#\{h \in [\mu - 1] : \sigma(h) > \sigma(\mu)\} = j, \tag{10}$$

thus

$$\#\{k \in [n] \setminus [\mu] : \sigma(k) < \sigma(\mu)\} = t - j. \tag{11}$$

We claim that if $\sigma \in \mathcal{C}_{P_\mu}^{(n)}(t)$ then

$$\max(0, \mu - n + t) \le j \le \min(t, \mu - 1);$$

indeed, if $j < \mu - n + t$, then

$$\#\{k \in [n] \setminus [\mu] : \sigma(k) < \sigma(\mu)\} = t - j > n - \mu$$

and this is impossible.

From (10) and (11) we have

$$\#\{k \in [n] : \sigma(k) < \sigma(\mu)\} = \mu - 1 - j + t - j,$$

hence

$$\sigma(\mu) = \mu - 2j + t,$$

and therefore

$$\begin{aligned} \mathcal{F}_{P_\mu}^{(n)}(t) &= \#\mathcal{C}_{P_\mu}^{(n)}(t) \\ &= (n-\mu)!\,(\mu-1)! \sum_{j=\max(0,\mu-n+t)}^{\min(t,\mu-1)} \binom{n-\sigma(\mu)}{j} \binom{\sigma(\mu)-1}{t-j} \end{aligned}$$

so we get (8).

Furthermore, taking into account the well-known formula

$$\sum_{k \ge 0} \binom{r+zk}{k} \binom{s-zk}{v-k} = \sum_{k \ge 0} \binom{r+s-k}{v-k} z^k, \tag{12}$$

where $r, s, v, z \in \mathbb{N}$, see e.g. Appendix A of [40], we have

$$\begin{aligned} \mathcal{F}_{P_\mu}^{(n)}(t) &= (n-\mu)!\,(\mu-1)! \sum_{j=\max(0,\mu-n+t)}^{\min(t,\mu-1)} \binom{n-\mu+2j-t}{j} \binom{\mu-2j+t-1}{t-j} \\ &\le (n-\mu)!\,(\mu-1)! \sum_{j=0}^{\mu-1} \binom{n-\mu+2j-t}{j} \binom{\mu-2j+t-1}{\mu-1-j} \\ &= (n-\mu)!\,(\mu-1)! \sum_{j=0}^{\mu-1} \binom{n-1-j}{\mu-1-j} 2^j \end{aligned}$$

since (12), and if the first sum runs over the full range $0, \ldots, \mu-1$, i.e. $\mu - 1 \le t \le n - \mu$, we have equality.

Therefore (9) follows. $\square$

Note that from (7) without loss of generality we can suppose $\mu \le \left\lceil \frac{n}{2} \right\rceil$, so there is always at least one t_0 such that $\mu - 1 \le t_0 \le n - \mu$, hence the inequality in (9) is sharp.

The following result follows immediately from Theorem 5.9.

Corollary 5.10. *For any $n \ge 2$,*

$$\begin{aligned} \Phi_{P_2}^{(n)}(X) &= \Phi_{Q_{2,2}}^{(n)}(X) = \Phi_{P_{n-1}}^{(n)}(X) \\ &= (n-2)!\left(1 + X^{n-1}\right) + (n+1)(n-2)! \sum_{j=1}^{n-2} X^j \end{aligned} \tag{13}$$

holds. $\square$

With just a mild change of $P_2 = Q_{2,2}$ it is possible to get $\mathcal{T} \subset T_n$ such that $\Phi_{\mathcal{T}}^{(n)}(X)$ is quite close to $\Phi_{P_2}^{(n)}(X) = \Phi_{Q_{2,2}}^{(n)}(X)$.

Theorem 5.11. *Let $n \geq 2$ and $\mathcal{T} = P_1 \biguplus (i, j)$ with $2 \leq i < j \leq n$. Then*

$$\Phi_{\mathcal{T}}^{(n)}(X) = \frac{(n-1)!}{2}(1 + X^n) + (n-1)!\sum_{j=1}^{n-1} X^j$$

holds.

Proof. If $\sigma \in \mathcal{C}_{\mathcal{T}}^{(n)}(0)$ then we have $\sigma(1) = 1$ and $2 \leq \sigma(i) < \sigma(j) \leq n$. Thus there are $\binom{n-1}{2}$ possibilities for $\{\sigma(i), \sigma(j)\}$ and $(n-3)!$ possibilities for $\{\sigma(k) : k \in [n] \setminus \{1, i, j\}\}$. Hence, from Proposition 3.2,

$$\mathcal{F}_{\mathcal{T}}^{(n)}(0) = \mathcal{F}_{\mathcal{T}}^{(n)}(n) = \frac{(n-1)!}{2}.$$

Let $k \in [n-1]$ and $\sigma \in \mathcal{C}_{\mathcal{T}}^{(n)}(k)$; then we have two mutually exclusive cases: $(i, j) \notin I_{\mathcal{T}}(\sigma)$ and $(i, j) \in I_{\mathcal{T}}(\sigma)$.

In the first case we have $\sigma(1) = k + 1$ and $\sigma(i), \sigma(j) \in [n] \setminus \{k + 1\}$ with $\sigma(i) < \sigma(j)$; whereas in the second case we have $\sigma(1) = k$ and $\sigma(i), \sigma(j) \in [n] \setminus \{k\}$ with $\sigma(i) > \sigma(j)$. So, as for $\mathcal{F}_{\mathcal{T}}^{(n)}(0)$, we have $\#\{\sigma \in \mathcal{C}_{\mathcal{T}}^{(n)}(k) : (i, j) \notin I_{\mathcal{T}}(\sigma)\} = \#\{\sigma \in \mathcal{C}_{\mathcal{T}}^{(n)}(k) : (i, j) \in I_{\mathcal{T}}(\sigma)\} = \frac{(n-1)!}{2}$, therefore $\mathcal{F}_{\mathcal{T}}^{(n)}(k) = (n-1)!$ as desired. $\square$

Corollary 5.12. *Let $n \geq 2$ and $\mathcal{T} = (i, j) \biguplus P_n$ with $1 \leq i < j \leq n-1$. Then*

$$\Phi_{\mathcal{T}}^{(n)}(X) = \frac{(n-1)!}{2}(1 + X^n) + (n-1)!\sum_{j=1}^{n-1} X^j$$

holds.

Proof. It is immediate from Theorem 5.11 and Proposition 3.15. $\square$

6. Unimodality and log-concavity

It is easy to verify that if $n = 2, 3$ then for any $\mathcal{T} \subset T_n$, $\Phi_{\mathcal{T}}^{(n)}(X)$ is log-concave and unimodal, so it is natural to wonder if this is true for any $n \geq 2$ and any $\mathcal{T} \subset T_n$. We prove that the answer is negative.

Note that if there exist $n_0 \geq 4$ and $\mathcal{T}_0 \subset T_{n_0}$ such that $\Phi_{\mathcal{T}_0}^{(n_0)}(X)$ is not log-concave or not unimodal then this holds for any $n \geq n_0$. In fact, $\mathcal{T}_0 \subset T_{n_0} \subset T_n$ and from Proposition 3.11 we have that also $\Phi_{\mathcal{T}_0}^{(n)}(X) = \frac{n!}{n_0!}\Phi_{\mathcal{T}}^{(n)}(X)$ is not log-concave or not unimodal. More exactly, for any $n \geq \operatorname{ord}(\mathcal{T}_0)$ and any $\mathcal{T} \subset T_n$ such that $\mathcal{T}_N = (\mathcal{T}_0)_N$ we have $\Phi_{\mathcal{T}}^{(n)}(X)$ is not log-concave or not unimodal.

Note also that for any $n \geq 2$ and any $\mathcal{T} \subset T_n$ from Proposition 3.1 if $\Phi_{\mathcal{T}}^{(n)}(X)$ is log-concave then it is also unimodal.

About unimodality, let $\mathcal{V}_n = \mathcal{E}_n \biguplus \{(1, n)\}$. Then if $n = 4$ we have that $\Phi_{\mathcal{V}_4}^{(4)}(X) = 1+8X+6X^2+8X^3+X^4$ is not unimodal (and not log-concave). Note that this does not hold for $n = 5$: in fact $\Phi_{\mathcal{V}_5}^{(5)}(X) = 1 + 22X + 37X^2 + 37X^3 + 22X^4 + X^5$ is log-concave (and unimodal). Instead, if $n = 5$ and $\mathcal{T} = \mathcal{V}_5 \biguplus \{(3, 5)\} \subset T_5$ we have that $\Phi_{\mathcal{T}}^{(5)}(X) = 1+15X+31X^2+26X^3+31X^4+15X^5+X^6$ is not unimodal (and not log-concave).

Open Problem 6.1. *Characterize $n \geq 4$ for which there exists $\mathcal{T} \subset T_n$ with $\operatorname{ord}(\mathcal{T}) = n$ such that $\Phi_{\mathcal{T}}^{(n)}(X)$ is not unimodal, and explicitly exhibit such $\mathcal{T}$.*

We feel that the answer is the following.

Conjecture 6.2. *For any $n \geq 4$ we define*

$$\mathcal{U}_n = \mathcal{E}_n \biguplus \{(1, n)\} \biguplus \{(1, k) : k \in [n-2] \setminus [2]\}.$$

Then $\Phi_{\mathcal{U}_n}^{(n)}(X)$ is not unimodal for any $n \geq 4$.

We have verified this for any $4 \leq n \leq 12$ using a computer.

Note that if the conjecture holds then from Proposition 3.15

$$(\mathcal{U}_n)_M = \mathcal{E}_n \biguplus \{(k, n) : k \in [n-2] \setminus \{2\}\}$$

is also a solution of the Open Problem 6.1.

Definition 6.3. For any $n \geq 2$, we define $\varphi_u(n)$ as the density of $\mathcal{T} \subset T_n$ with $\operatorname{ord}(\mathcal{T}) = n$ such that $\Phi_{\mathcal{T}}^{(n)}(X)$ is not unimodal, viz.

$$\varphi_u(n) = \frac{\#\{\mathcal{T} \subset T_n : \operatorname{ord}(\mathcal{T}) = n, \quad \text{and } \Phi_{\mathcal{T}}^{(n)}(X) \quad \text{is not unimodal}\}}{\#\{\mathcal{T} \subset T_n : \operatorname{ord}(\mathcal{T}) = n\}}.$$

Open Question 6.4. *Is it true that*

$$\limsup_{n\to\infty} \varphi_u(n) = 0,$$

(viz. $\lim_{n\to\infty} \varphi_u(n)$ *exists and it is equal to* 0*)?*

About log-concavity, for any $n \geq 4$ we are able to find some $\mathcal{T} \subset T_n$ with $\mathrm{ord}(\mathcal{T}) = n$ such that $\Phi_{\mathcal{T}}^{(n)}(X)$ are not log-concave.

Theorem 6.5. *Let* $n \geq 3$ *and*

$$\mathcal{T} = \{(1,k) : k \in [n] \setminus [2]\} \biguplus \{(2,k) : k \in [n] \setminus [2]\},$$
$$\mathcal{U} = \{(k, n-1) : k \in [n-2]\} \biguplus \{(k,n) : k \in [n-2]\}.$$

Then

$$\Phi_{\mathcal{T}}^{(n)}(X) = \Phi_{\mathcal{U}}^{(n)}(X) = 2(n-2)! \sum_{j=0}^{n-3} \left(1 + \left\lfloor \frac{j}{2} \right\rfloor\right) \left[X^j + X^{2(n-2)-j}\right] + 2(n-2)! \left(1 + \left\lfloor \frac{n-2}{2} \right\rfloor\right) X^{n-2}$$

holds.

In particular, if $n \geq 4$ *then* $\Phi_{\mathcal{T}}^{(n)}(X) = \Phi_{\mathcal{U}}^{(n)}(X)$ *is not log-concave.*

Proof. Note that $\mathcal{T}_M = \mathcal{U}$, so from Proposition 3.15 $\Phi_{\mathcal{T}}^{(n)}(X) = \Phi_{\mathcal{U}}^{(n)}(X)$.

We remark that for any $j = 0, \ldots, \#\mathcal{T}$, if $\sigma \in \mathcal{C}_{\mathcal{T}}^{(n)}(j)$ then any $\tau \in S_n$ such that

$$\{\sigma(1), \sigma(2)\} = \{\tau(1), \tau(2)\}$$

belongs to $\mathcal{C}_{\mathcal{T}}^{(n)}(j)$, so the factor $2(n-2)!$ is clear. Thus in order to study $\sigma \in \mathcal{C}_{\mathcal{T}}^{(n)}(j)$ we are allowed to study only $\sigma(1), \sigma(2)$ and without loss of generality we can suppose $1 \leq \sigma(1) < \sigma(2) \leq n$.

Furthermore $\#\mathcal{T} = 2(n-2)$, thus $\mathcal{F}_{\mathcal{T}}^{(n)}(2n-4-j) = \mathcal{F}_{\mathcal{T}}^{(n)}(j)$ for any $j = 0, \ldots, 2(n-2)$ from Proposition 3.2, so in order to compute $\mathcal{F}_{\mathcal{T}}^{(n)}(j)$ we can suppose without loss of generality $0 \leq j \leq n-2$.

For any $0 \leq j \leq n-2$ let

$$\sigma \in \mathcal{Z}_j = \{\tau \in \mathcal{C}_{\mathcal{T}}^{(n)}(j) : \tau(1) < \tau(2)\}.$$

We call

$$\mu = \#\{\sigma(t) : t \in [n] \setminus [2], \sigma(t) < \sigma(1)\}$$

and

$$\nu = \#\{\sigma(t) : t \in [n] \setminus [2], \sigma(t) < \sigma(2)\}.$$

Then from our assumptions on σ we have

$$j = \mu + \nu \quad \text{with } 0 \leq \mu \leq \nu,$$

hence

$$0 \leq \mu \leq \left\lfloor \frac{j}{2} \right\rfloor. \tag{14}$$

It is easy to see that this implies

$$\sigma(1) = \mu + 1 \quad \text{and} \quad \sigma(2) = \nu + 2 = j - \mu + 2, \tag{15}$$

and on the other hand if $0 \leq j \leq n-2$, any $\sigma \in S_n$ such that $(\sigma(1), \sigma(2))$ satisfies (15) and (14) belongs to $\mathcal{Z}_j$.

Therefore for any $0 \leq j \leq n-2$

$$\#\{(\sigma(1), \sigma(2)) : \sigma \in \mathcal{Z}_j\} = 1 + \left\lfloor \frac{j}{2} \right\rfloor$$

and the desired result follows.

The statement about the log-concavity is obvious. $\square$

Note that $\mathcal{T} = \mathcal{U}$, i.e. we are considering a self-mirrored set, if and only if $n = 4$.

Note also that it is not necessary to explicitly compute $\Phi_{\mathcal{T}}^{(n)}(X)$ in order to prove that it is not log-concave. In the following results we generalize the class of $\mathcal{T}$ such that $\Phi_{\mathcal{T}}^{(n)}(X)$ is not log-concave.

Theorem 6.6. *Let $n \geq 4$, $2 \leq r \leq n-2$ and*

$$\mathcal{T}_r = \bigcup_{j=1}^{r} \{(j,k) : k \in [n] \setminus [r]\}.$$

Then $\Phi^{(n)}_{\mathcal{T}_r}(X)$ is not log-concave.

Proof. For any $\sigma \in S_n$ we define

$$A(\sigma) = \{\sigma(j) : j \in [r]\}.$$

Note that for any $j = 0, \ldots, \#\mathcal{T}_r$ if $\sigma \in \mathcal{C}^{(n)}_{\mathcal{T}_r}(j)$ then for any $\tau \in S_n$ such that

$$A(\tau) = A(\sigma)$$

we have $\tau \in \mathcal{C}^{(n)}_{\mathcal{T}_r}(j)$, therefore $\mathcal{F}^{(n)}_{\mathcal{T}_r}(j) = \alpha_j (n-r)!r!$ with $\alpha_j \in \mathbb{N} \setminus \{0\}$.
We prove that $1 = \alpha_0 = \alpha_1 < \alpha_2 = 2$, viz.

$$0 < \mathcal{F}^{(n)}_{\mathcal{T}_r}(0) = \mathcal{F}^{(n)}_{\mathcal{T}_r}(1) = r!(n-r)! < \mathcal{F}^{(n)}_{\mathcal{T}_r}(2) = 2(n-r)!r!,$$

so the desired result follows.
We have that $\sigma \in \mathcal{C}^{(n)}_{\mathcal{T}_r}(0)$ if and only if

$$A(\sigma) = [r]$$

and $\sigma \in \mathcal{C}^{(n)}_{\mathcal{T}_r}(1)$ if and only if

$$A(\sigma) = [r-1] \biguplus \{r+1\}.$$

On the other hand, $\sigma \in \mathcal{C}^{(n)}_{\mathcal{T}_r}(2)$ if and only if either

$$A(\sigma) = [r-1] \biguplus \{r+2\}$$

or

$$A(\sigma) = [r-2] \biguplus \{r, r+1\}. \quad \square$$

Note that if $r \in \{2, n-2\}$ we have the sets of Theorem 6.5, for which we computed $\Phi^{(n)}_{\mathcal{T}}(X)$.
In the next Proposition we find one more $\mathcal{T}$ such that $\Phi^{(n)}_{\mathcal{T}}(X)$ is not log-concave.

Proposition 6.7. *For any $n \geq 4$ and any $1 \leq \mu \leq n-3$ let*

$$\mathcal{T}_\mu = \{(j, \mu+1) : j \in [\mu]\} \biguplus \{(j, \mu+2) : j \in [\mu]\} \biguplus \{(\mu+1, k) : k \in [n] \setminus [\mu+2]\} \biguplus \{(\mu+2, k) : k \in [n] \setminus [\mu+2]\}.$$

Then $\Phi^{(n)}_{\mathcal{T}_\mu}(X)$ is not log-concave.

Proof. We prove that

$$\begin{aligned} &2(n-(\mu+2))!\mu! = \mathcal{F}^{(n)}_{\mathcal{T}_\mu}(0) \\ &< 4(n-(\mu+2))!\mu! = \mathcal{F}^{(n)}_{\mathcal{T}_\mu}(1) \\ &< 12(n-(\mu+2))!\mu! \leq \mathcal{F}^{(n)}_{\mathcal{T}_\mu}(2), \end{aligned} \tag{16}$$

so the desired result follows.
For any $\sigma \in S_n$ we define

$$A(\sigma) = \{\sigma(j) : j \in [\mu]\},$$
$$B(\sigma) = \{\sigma(\mu+1), \sigma(\mu+2)\}.$$

We have that $\sigma \in \mathcal{C}^{(n)}_{\mathcal{T}_\mu}(0)$ if and only if

$$\begin{cases} A(\sigma) = [\mu] \\ B(\sigma) = \{\mu+1, \mu+2\}, \end{cases}$$

thus we get the first row of (16).
$\sigma \in \mathcal{C}^{(n)}_{\mathcal{T}_\mu}(1)$ if and only if either

$$\begin{cases} A(\sigma) = [\mu-1] \biguplus \{\mu+1\} \\ B(\sigma) = \{\mu, \mu+2\} \end{cases}$$

or

$$\begin{cases} A(\sigma) = [\mu] \\ B(\sigma) = \{\mu + 1, \mu + 3\}, \end{cases}$$

thus we get the second row of (16).

Now we consider any $\sigma \in S_n$ such that either

$$\begin{cases} A(\sigma) = [\mu - 1] \biguplus \{\mu + 2\} \\ B(\sigma) = \{\mu, \mu + 1\} \end{cases}$$

or

$$\begin{cases} A(\sigma) = [\mu] \\ B(\sigma) = \{\mu + 2, \mu + 3\} \end{cases}$$

or

$$\begin{cases} A(\sigma) = [\mu - 1] \biguplus \{\mu + 1\} \\ B(\sigma) = \{\mu, \mu + 3\} \end{cases}$$

or

$$\begin{cases} A(\sigma) = [\mu - 1] \biguplus \{\mu + 3\} \\ B(\sigma) = \{\mu, \mu + 1\} \end{cases}$$

or

$$\begin{cases} A(\sigma) = [\mu - 1] \biguplus \{\mu + 1\} \\ B(\sigma) = \{\mu + 2, \mu + 3\} \end{cases}$$

or

$$\begin{cases} A(\sigma) = [\mu - 1] \biguplus \{\mu + 2\} \\ B(\sigma) = \{\mu, \mu + 3\}. \end{cases}$$

Then we have that $\sigma \in \mathcal{C}^{(n)}_{\mathcal{T}_\mu}(2)$, thus we get also the third row of (16), and the desired result follows. $\square$

7. Remarks

We note it is possible to define an analogue of the polynomial $\Phi^{(n)}_{\mathcal{T}}(X)$ in every finite Coxeter system (W, S) (see [12,17,22,26,39,42,51]), substituting the function inv with the function *length*, and the set T_n with the reflection set of W.

More exactly, given a finite Coxeter system (W, S), where W is a finite Coxeter group with distinguished generating set S, and $u \in W$, we denote by $l(u)$ the length of u in W, with respect to S. We denote

$$D(u) = \{s \in S : l(us) < l(u)\}$$

the *descent set* of $u \in W$, and

$$T = \{usu^{-1} : s \in S,\ u \in W\}$$

the *reflection set* of W.

Let $\mathcal{T} \subset T$: we define

$$d_{\mathcal{T}}(u) = \#\{z \in \mathcal{T} : l(uz) < l(u)\}$$

for any $u \in W$, and

$$\Phi^W_{\mathcal{T}}(X) = \sum_{u \in W} X^{d_{\mathcal{T}}(u)} = \sum_{j=0}^{\#\mathcal{T}} \mathcal{F}^W_{\mathcal{T}}(j) X^j \in \mathbb{N}[X],$$

where $\mathcal{F}^W_{\mathcal{T}}(j) = \#\{u \in W : d_{\mathcal{T}}(u) = j\}$.

Note that if $\mathcal{T} = T$ then $\Phi^W_T(X)$ is the *Poincaré polynomial* of (W, S), and if $\mathcal{T} = S$ then $\Phi^W_S(X)$ is the *Eulerian polynomial* of (W, S), see e.g. [20,23,46,54].

We also note that in the case of an infinite Coxeter system (W, S) with finite rank, the Eulerian distribution $\sum_{u \in W} X^{\mathrm{des}(u)}$ does not make sense as a formal power series in X, since there are only finitely many values $\{0, \ldots, \#S\}$ of $\mathrm{des}(u)$ and hence infinitely many group elements u with the same value of $\mathrm{des}(u)$.

In [25] the author investigates the polynomials $\Phi^W_{\mathcal{T}}(X)$ for any generic finite Coxeter system (W, S) and any $\mathcal{T} \subset T$.

We remark that this combinatorial approach can be very effective for studying a lot of algebraic and combinatorial topics about a generic finite Coxeter system, such as representations, see [2], orderings, see [8,9,11], descent classes, parabolic subgroups and generalized quotients, see [13,14].

Acknowledgements

The author would like to thank his advisor Francesco Brenti for suggesting this problem and his helpful advice.

This work is part of the Ph. D. thesis of the author at Università degli Studi di Roma "Tor Vergata", and it was partially supported by European Research Training Network "Algebraic Combinatorics in Europe" under the grant # HPRN-CT-2001-00272.

References

[1] R.M. Adin, F. Brenti, Y. Roichman, Descent numbers and major indices for the hyperoctahedral group, Adv. Appl. Math. 27 (2001) 210–224.
[2] R. Adin, F. Brenti, Y. Roichman, A unified construction of Coxeter group representations (I), Adv. in Appl. Math. 37 (2006) 31–67.
[3] R. Adin, F. Brenti, Y. Roichman, Equi–distribution over descent classes of the hyperoctahedral group, J. Combin. Theory Ser. A 113 (2006) 917–933.
[4] R.M. Adin, Y. Roichman, The flag major index and group actions on polynomial rings, European J. Combin. 22 (2001) 431–446.
[5] I. Anderson, Combinatorics of Finite Sets, Dover Publ., New York, 2002.
[6] E.A. Bender, Asymptotic methods in enumeration, SIAM Rev. 16 (1974) 485–515.
[7] E.A. Bender, Errata: Asymptotic methods in enumeration, SIAM Rev. 16 (1974) 485–515; SIAM Rev. 18 (1976) 292.
[8] C.D. Bennett, R.J. Blok, Partial orders generalizing the weak order on Coxeter groups, J. Combin Theory Ser. A 102 (2003) 331–346.
[9] C.D. Bennett, L. Evani, D. Grabiner, A simple definitions for the universal Grassmannian order, J. Combin Theory Ser. A 102 (2003) 347–366.
[10] R. Biagioli, Major and descent statistics for the even-signed permutation group, Adv. Appl. Math. 31 (2003) 163–179.
[11] A. Björner, Orderings of Coxeter groups, Contemp. Math. 34 (1984) 175–195.
[12] A. Björner, F. Brenti, Combinatorics of Coxeter groups, in: Graduate Texts in Mathematics, vol. 231, Springer-Verlag, New York, 2005.
[13] A. Björner, M.L. Wachs, Generalized quotients in Coxeter groups, Trans. Amer. Math. Soc. 308 (1988) 1–37.
[14] A. Björner, M.L. Wachs, Permutation statistics and linear extensions of posets, J. Combin. Theory Ser. A 58 (1991) 85–114.
[15] M. Bóna, A simplicial complex of 2-stack sortable permutations, Adv. Appl. Math. 29 (2002) 499–508.
[16] M. Bóna, R. Ehrenborg, A combinatorial proof of the log-concavity of the numbers of permutations with k runs, J. Combin. Theory Ser. A 90 (2000) 293–303.
[17] N. Bourbaki, Groupes et Algèbres de Lie, in: Éléments de mathématique, Fascicule XXXIV, Hermann, Paris, 1968 (Chapitres IV–VI, reprinted in Masson, Paris 1981).
[18] F. Brenti, Unimodal polynomials arising from symmetric functions, Proc. Amer. Math. Soc. 108 (1990) 1133–1141.
[19] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: An update, Contemp. Math. 178 (1994) 71–89.
[20] F. Brenti, q-Eulerian polynomials arising from coxeter groups, European J. Combin. 15 (1994) 417–441.
[21] F. Brenti, Hilbert polynomials in combinatorics, J. Algebraic Combin. 7 (1998) 127–156.
[22] K.S. Brown, Buildings, Springer-Verlag, New York, 1989.
[23] C.-O. Chow, On the Eulerian polynomials of type D, European J. Combin. 24 (2003) 391–408.
[24] L. Comtet, Advanced Combinatorics, D. Reidel, Dordrecht, 1974.
[25] A. Conflitti, Enumeration by associated reflections on Coxeter systems, J. Algebra (in press).
[26] V.V. Deodhar, Some characterizations of Coxeter groups, Enseign. Math. 32 (1986) 111–120.
[27] R. Ehrenborg, M. Levin, M.A. Readdy, A probabilistic approach to the descent statistic, J. Combin. Theory Ser. A 98 (2002) 150–162.
[28] R. Ehrenborg, E. Steingrímsson, The excedence set of a permutation, Adv. Appl. Math. 24 (2000) 284–299.
[29] R. Ehrenborg, E. Steingrímsson, Yet another triangle for the Genocchi numbers, European J. Combin. 21 (2000) 593–600.
[30] D. Foata, On the Netto inversion number of a sequence, Proc. Amer. Math. Soc. 19 (1968) 236–240.
[31] D. Foata, M.-P. Schützenberger, Major index and inversion number of permutations, Math. Nachr. 83 (1978) 143–159.
[32] D. Foata, M.-P. Schützenberger, Théorie Géométrique des Polynômes Eulériens, Springer-Verlag, Berlin, 1970.
[33] J.C. Fu, W.Y.W. Lou, Y.-J. Wang, On the exact distribution of Eulerian and Simon Newcomb numbers associated with random permutations, Statist. Probab. Lett. 42 (1999) 115–125.
[34] V. Gasharov, On the Neggers–Stanley conjecture and the Eulerian polynomials, J. Combin. Theory Ser. A 82 (1998) 134–146.
[35] V. Gasharov, Factoring the Poincaré polynomials for the Bruhat order on S_n, J. Combin. Theory Ser. A 83 (1998) 159–164.
[36] R.L. Graham, D.E. Knuth, O. Patashnik, Concrete Mathematics, 2nd ed., Addison-Wesley Publishing Company, Reading, MA, 1994.
[37] C. Greene, D.J. Kleitman, Proofs techniques in the theory of finite sets, in: Studies in Combinatorics, in: MAA Studies in Mathematics, vol. 17, The Mathematical Association of America, 1978, pp. 22–79.
[38] B. Harris, C.J. Park, A generalization of the Eulerian numbers with a probabilistic application, Statist. Probab. Lett. 20 (1994) 37–47.
[39] H. Hiller, Geometry of Coxeter Groups, Pitman (Advanced Publishing Program), Boston, London, Melbourne, 1982.
[40] M. Hofri, Probabilistic Analysis of Algorithms, Springer-Verlag, New York, 1987.
[41] S.G. Hoggar, Chromatic polynomials and logarithmic concavity, J. Combinatorial Theory Ser. B 16 (1974) 248–254.
[42] J.H. Humphreys, Reflection Groups and Coxeter Groups, Cambridge University Press, Cambridge, 1990.
[43] P.A. MacMahon, The indices of permutations and the derivation therefrom of functions of a single variable associated with the permutations of any assemblage objects, Amer. J. Math. 35 (1913) 281–322.
[44] P.A. MacMahon, Combinatory Analysis, Chelsea Publishing Company, New York, 1984.
[45] J.-L. Nicolas, Propriétés arithmétiques de certains nombres eulériens, Colloq. Math. 65 (1993) 307–320.
[46] V. Reiner, The distribution of descent and length in a Coxeter group, Electron. J. Combin. 2 (1995), Research paper 25, 20 pp..
[47] L. Solomon, Presenting the symmetric group with transpositions, J. Algebra 168 (1994) 521–524.
[48] R.P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics and geometry, in: Graph Theory and Its Applications: East and West, in: Annals of the New York Academy of Sciences, vol. 576, 1989, pp. 500–534.
[49] R.P. Stanley, Enumerative Combinatorics Vol. 1, Cambridge University Press, 1997.
[50] J.R. Stembridge, Eulerian numbers, tableaux, and the Betti numbers of a toric variety, Discrete Math. 99 (1992) 307–320.
[51] R. Suter, Coxetergruppen, Elem. Math. 52 (1997) 12–22 (in German).
[52] S. Tanimoto, An operator on permutations and its applications to Eulerian numbers, European J. Combin. 22 (2001) 569–576.
[53] S. Tanimoto, A study of Eulerian numbers by means of an operator on permutations, European J. Combin. 24 (2003) 33–43.
[54] R. Winkel, A combinatorial derivation of the Poincaré polynomials of the finite irreducible Coxeter groups, Discrete Math. 239 (2001) 83–99.
[55] J. Worpitzky, Studien über die Bernoullischen und Eulerschen Zahlen, J. Reine Angew. Math. 94 (1883) 203–232.